FINITE ELEMENT APPROXIMATIONS FOR A LINEAR CAHN-HILLIARD-COOK EQUATION DRIVEN BY THE SPACE DERIVATIVE OF A SPACE-TIME WHITE NOISE

GEORGIOS T. KOSSIORIS[‡] AND GEORGIOS E. ZOURARIS[‡]

Abstract. We consider an initial- and Dirichlet boundary- value problem for a linear Cahn-Hilliard-Cook equation, in one space dimension, forced by the space derivative of a space-time white noise. First, we propose an approximate regularized stochastic parabolic problem discretizing the noise using linear splines. Then fully-discrete approximations to the solution of the regularized problem are constructed using, for the discretization in space, a Galerkin finite element method based on H^2-piecewise polynomials, and, for time-stepping, the Backward Euler method. Finally, we derive strong a priori estimates for the modeling error and for the numerical approximation error to the solution of the regularized problem.

1. Introduction

Let $T>0$, $D=(0,1)$ and $(\Omega,\mathcal{F},P)$ be a complete probability space. Then we consider the following model initial- and Dirichlet boundary- value problem for a linear Cahn-Hilliard-Cook equation: find a stochastic function $u:[0,T]\times\overline{D}\to\mathbb{R}$ such that

$$\begin{aligned}\partial_t u+\partial_x^4 u+\mu\,\partial_x^2 u&=\partial_x\dot{W}(t,x)\quad\forall\,(t,x)\in(0,T]\times D,\\ \partial_x^{2m}u(t,\cdot)\big|_{\partial D}&=0\quad\forall\,t\in(0,T],\quad m=0,1,\\ u(0,x)&=0\quad\forall\,x\in D,\end{aligned}\tag{1.1}$$

a.s. in Ω, where $\dot{W}$ denotes a space-time white noise on $[0,T]\times D$ (see, e.g., [23], [11]) and μ is a real constant for which there exists $\kappa\in\mathbb{N}$ such that

$$(\kappa-1)^2\,\pi^2\le\mu<\kappa^2\,\pi^2,\tag{1.2}$$

where $\mathbb{N}$ is the set of all positive integers. The above stochastic partial differential equation combines two independent characteristics. On the one hand it corresponds to the linearization of the Cahn-Hilliard-Cook equation around a homogeneous initial state, in the spinodal region, that governs the dynamics of spinodal decomposition in metal alloys; see e.g. [4], and references therein. On the other hand the forcing noise is a derivative of a space-time white noise that physically arises in generalized Cahn-Hilliard equations, which are equations of conservative type describing the evolution of an order parameter in phase transitions (see [10]; cf. [12], [2], [19]).

The mild solution of the problem above (cf. [6]) is given by the formula

$$u(t,x)=\int_0^t\int_D\Psi(t-s;x,y)\,dW(s,y),\tag{1.3}$$

where

$$\Psi(t;x,y)=-\sum_{k=1}^{\infty}e^{-\lambda_k^2\,(\lambda_k^2-\mu)t}\,\varepsilon_k(x)\,\varepsilon_k'(y)\quad\forall\,(t,x,y)\in(0,T]\times\overline{D}\times\overline{D},\tag{1.4}$$

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[‡]Department of Mathematics, University of Crete, GR–714 09 Heraklion, Crete, Greece.

with $\lambda_k := k\,\pi$ for $k \in \mathbb{N}$, and $\varepsilon_k(z) := \sqrt{2}\,\sin(\lambda_k\,z)$ for $z \in \overline{D}$ and $k \in \mathbb{N}$. Observe that $\Psi(t;x,y) = -\partial_y G(t;x,y)$, where $G(t;x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k^2\,(\lambda_k^2-\mu)t}\,\varepsilon_k(x)\,\varepsilon_k(y)$ for all $(t,x,y) \in (0,T] \times \overline{D} \times \overline{D}$, is the space-time Green kernel of the corresponding deterministic parabolic problem: find a deterministic function $w : [0,T] \times \overline{D} \to \mathbb{R}$ such that

$$
\begin{aligned}
\partial_t w + \partial_x^4 w + \mu\,\partial_x^2 w = 0 \quad \forall\,(t,x) \in (0,T] \times D,\\
\partial_x^{2m} w(t,\cdot)\big|_{\partial D} = 0 \quad \forall\,t \in (0,T], \quad m = 0,1,\\
w(0,x) = w_0(x) \quad \forall\,x \in D.
\end{aligned}
\tag{1.5}
$$

The goal of the paper at hand is to propose and analyze a methodology of constructing finite element approximations to u.

1.1. **The regularized problem.** Our first step is to construct below an approximate to (1.1) regularized problem getting inspiration from the work [1] for the stochastic heat equation with additive space-time white noise (cf. [14], [15]).

Let $N_\star \in \mathbb{N}$, $\Delta t := \frac{T}{N_\star}$, $J_\star \in \mathbb{N}$ and $\Delta x := \frac{1}{J_\star}$. Then, consider a partition of the interval $[0,T]$ with nodes $(t_n)_{n=0}^{N_\star}$ and a partition of $\overline{D}$ with nodes $(x_j)_{j=0}^{J_\star}$, given by $t_n := n\,\Delta t$ for $n = 0,\dots,N_\star$ and $x_j := j\,\Delta x$ for $j = 0,\dots,J_\star$. Also, set $T_n := (t_{n-1},t_n)$ for $n = 1,\dots,N_\star$, and $D_j := (x_{j-1},x_j)$ for $j = 1,\dots,J_\star$.

First, we let $\mathcal{S}_\star$ be the space of functions which are continuous on $\overline{D}$ and piecewise linear over the above specified partition of D, i.e.,

$$
\mathcal{S}_\star := \left\{ s \in C(\overline{D};\mathbb{R}) : \quad s\big|_{D_j} \in \mathbb{P}^1(D_j) \ \text{ for } \ j = 1,\dots,J_\star \right\} \subset H^1(D).
$$

It is well-known that $\dim(\mathcal{S}_\star) = J_\star + 1$ and that the functions $(\psi_i)_{i=1}^{J_\star+1} \subset \mathcal{S}_\star$ defined by:

$$
\begin{gathered}
\psi_1(x) := \tfrac{1}{\Delta x}\,(x_1 - x)^+, \quad \psi_{J_\star+1}(x) := \tfrac{1}{\Delta x}\,(x - x_{J_\star-1})^+,\\
\psi_i(x) := \tfrac{1}{\Delta x}\,\left[\,(x - x_{i-2})\,\mathcal{X}_{(x_{i-2},x_{i-1}]} + (x - x_i)\,\mathcal{X}_{(x_{i-1},x_i]}\,\right], \quad i = 2,\dots,J_\star,
\end{gathered}
$$

consist the well-known *hat functions* basis of $\mathcal{S}_\star$, where, for any $A \subset \mathbb{R}$, by $\mathcal{X}_A$ we denote the index function of A. Next, consider the fourth-order linear stochastic parabolic problem:

$$
\begin{aligned}
\partial_t \widehat{u} + \partial_x^4 \widehat{u} + \mu\,\partial_x^2 \widehat{u} = \partial_x \widehat{W} \quad \text{in} \quad (0,T] \times D,\\
\partial_x^{2m} \widehat{u}(t,\cdot)\big|_{\partial D} = 0 \quad \forall\,t \in (0,T], \quad m = 0,1,\\
\widehat{u}(0,x) = 0 \quad \forall\,x \in D,
\end{aligned}
\tag{1.6}
$$

a.e. in Ω, where:

$$
\widehat{W}(t,x) := \tfrac{1}{\Delta t}\,\sum_{n=1}^{N_\star} \mathcal{X}_{T_n}(t)\,\left[\sum_{\ell=1}^{J_\star+1}\left(\sum_{m=1}^{J_\star+1} G_{\ell,m}^{-1}\,R_{n,m}\right)\psi_\ell(x)\right], \quad \forall\,(t,x) \in [0,T] \times \overline{D},
$$

G is a real, $(J_\star+1) \times (J_\star+1)$, symmetric and positive definite matrix with

$$
G_{i,j} := (\psi_j, \psi_i)_{0,D}, \quad i,\,j = 1,\dots,J_\star+1,
$$

and

$$
R_{n,i} := \int_{T_n}\int_D \psi_i(x)\,dW(t,x), \quad i = 1,\dots,J_\star+1, \quad n = 1,\dots,N_\star.
$$

The solution of the problem (1.6), has the integral representation (see, e.g., [17])

$$
\begin{aligned}
\widehat{u}(x,t) &= \int_0^t\!\!\int_D G(t-s;x,y)\,\partial_y \widehat{W}(s,y)\,dsdy\\
&= \int_0^t\!\!\int_D \Psi(t-s;x,y)\,\widehat{W}(s,y)\,dsdy, \quad \forall\,(t,x) \in [0,T] \times \overline{D}.
\end{aligned}
\tag{1.7}
$$

Remark 1.1. *A simple computation verifies that G is a tridiagonal matrix with $G_{1,1} = G_{J_\star+1,J_\star+1} = \frac{\Delta x}{3}$, $G_{i,i} = \frac{2\,\Delta x}{3}$ for $i = 2,\dots,J_\star$, and $G_{i,i+1} = \frac{\Delta x}{6}$ for $i = 1,\dots,J_\star$. Since G is symmetric we have in addition that $G_{i-1,i} = \frac{\Delta x}{6}$ for $i = 2,\dots,J_\star+1$.*

Remark 1.2. *Let $\mathcal{I} = \{(n,i) : n = 1,\dots,N_\star,\ i = 1,\dots,J_\star+1\}$. Using the properties of the stochastic integral (see, e.g., [23]), we conclude that $R_{n,i} \sim \mathcal{N}(0,\Delta t\, G_{i,i})$ for all $(n,i) \in \mathcal{I}$. Also, we observe that $\mathbb{E}[R_{n,i}\,R_{n',j}] = 0$ for (n,i), $(n',j) \in \mathcal{I}$ with $n \neq n'$, and hence they are independent since they are Gaussian. In addition, we have that $\mathbb{E}[R_{n,i}\,R_{n,j}] = \Delta t\, G_{i,j}$ for (n,i), $(n,j) \in \mathcal{I}$. Thus, for a given n the random variables $(R_{n,i})_{i=1}^{J_\star+1}$ are Gaussian and correlated, with correlation matrix $\Delta t\, G$.*

1.2. **The numerical method.** Our second step is to construct finite element approximations of the solution $\widehat{u}$ to the regularized problem.

Let $M \in \mathbb{N}$, $\Delta\tau := \frac{T}{M}$, $\tau_m := m\,\Delta\tau$ for $m = 0,\dots,M$, and $\Delta_m := (\tau_{m-1},\tau_m)$ for $m = 1,\dots,M$. Also, let $r \in \{2,3\}$, and $M_h^r \subset H^2(D) \cap H_0^1(D)$ be a finite element space consisting of functions which are piecewise polynomials of degree at most r over a partition of D in intervals with maximum mesh-length h. Then, computable fully-discrete approximations of $\widehat{u}$ are constructed by using the Backward Euler finite element method, which first sets

$$\widehat{U}_h^0 := 0 \tag{1.8}$$

and then, for $m = 1,\dots,M$, finds $\widehat{U}_h^m \in M_h^r$ such that

$$(\widehat{U}_h^m - \widehat{U}_h^{m-1},\chi)_{0,D} + \Delta\tau\left[\,((\widehat{U}_h^m)'',\chi'')_{0,D} + \mu\,((\widehat{U}_h^m)'',\chi)_{0,D}\right] = \int_{\Delta_m} (\partial_x\widehat{W},\chi)_{0,D}\,d\tau \tag{1.9}$$

for all $\chi \in M_h^r$, where $(\cdot,\cdot)_{0,D}$ is the usual $L^2(D)-$inner product.

1.3. **An overview of the paper and related references.** Our analysis first focus on the estimation of the modeling error, i.e. the difference $u - \widehat{u}$, in terms of the discretization parameters Δt and Δx. Indeed, working with the integral representation of u and $\widehat{u}$, we obtain (see Theorem 3.1)

$$\max_{t\in[0,T]}\left\{\int_\Omega\left(\int_D |u(t,x)-\widehat{u}(t,x)|^2\,dx\right)dP\right\}^{\frac12} \leq C_{\rm me}\left(\epsilon^{-\frac12}\,\Delta x^{\frac12-\epsilon} + \Delta t^{\frac18}\right),\quad \forall\,\epsilon\in(0,\tfrac12], \tag{1.10}$$

where $C_{\rm me}$ is a positive constant that is independent of Δx, Δt and ϵ. Next target in our analysis, is to provide the fully discrete approximations of $\widehat{u}$ defined in Section 1.2 with a convergence result, which is achieved by proving the following strong error estimate (see Theorem 5.3)

$$\max_{0\leq m\leq M}\left\{\int_\Omega\left(\int_D \left|\widehat{U}_h^m(x)-\widehat{u}(\tau_m,x)\right|^2 dx\right)dP\right\}^{\frac12} \leq C_{\rm ne}\left(\epsilon_1^{-\frac12}\,\Delta\tau^{\frac18-\epsilon_1} + \epsilon_2^{-\frac12}\,h^{\nu(r)-\epsilon_2}\right), \tag{1.11}$$

for all $\epsilon_1 \in (0,\frac18]$ and $\epsilon_2 \in (0,\nu(r)]$ with $\nu(2) = \frac13$ and $\nu(3) = \frac12$, where $C_{\rm ne}$ is a positive constant independent of ϵ_1, ϵ_2, $\Delta\tau$, h, Δx and Δt. To get the error estimate (1.11) we use as an auxilliary tool the Backward-Euler time-discrete approximations of $\widehat{u}$ which are defined in Section 4. Thus, we can see the numerical approximation error as a sum of two types of error: the *time-discretization* error and the *space-discretization* error. The *time-discretization* error is the approximation error of the Backward Euler time-discrete approximations which is estimated in Theorem 4.2, while the *space-discretization* error is the error of approximating the Backward Euler time-discrete approximations by the Backward Euler finite element approximations, which is estimated in Proposition 5.2.

Let us expose some related bibliography. The work [18] contains a general convergence analysis for a class of time-discrete approximations to the solution of stochastic parabolic problems, the assumptions of which may cover problem (1.1). However, the approach we adopt here is different since first we introduce a space-time discretization of the noise and then we analyze time-discrete approximations to the solution. We would like to note that we are not aware of another work providing a rigorous convergence analysis for fully discrete finite element approximations to a stochastic parabolic equation forced by the space derivative of a space-time white noise. We refer the reader to our previous work [14], [15] and to [16] for the construction and the convergence analysis of Backward Euler finite element approximations of the solution to the problem (1.1) when $\mu = 0$ and an additive space-time white noise $\dot{W}$ is forced instead of

$\partial_x \dot{W}$. Finally, we refer the reader to [8], [1], [13], [3], [22] and [24] for the analysis of the finite element method for second order stochastic parabolic problems forced by an additive space-time white noise.

We close the section by an overview of the paper. Section 2 introduces notation, and recalls or proves several results often used in the paper. Section 3 is dedicated to the estimation of the modeling error. Section 4 defines the Backward Euler time-discrete approximations of $\widehat{u}$ and analyzes its convergence. Section 5 contains the error analysis for the Backward Euler fully-discrete approximations of $\widehat{u}$.

2. Notation and Preliminaries

2.1. **Function spaces and operators.** Let $I \subset \mathbb{R}$ be a bounded interval. We denote by $L^2(I)$ the space of the Lebesgue measurable functions which are square integrable on I with respect to Lebesgue's measure dx, provided with the standard norm $\|g\|_{0,I} := \left(\int_I |g(x)|^2\, dx\right)^{\frac{1}{2}}$ for $g \in L^2(I)$. The standard inner product in $L^2(I)$ that produces the norm $\|\cdot\|_{0,I}$ is written as $(\cdot,\cdot)_{0,I}$, i.e., $(g_1, g_2)_{0,I} := \int_I g_1(x) g_2(x)\, dx$ for g_1, $g_2 \in L^2(I)$. Let $\mathbb{N}_0$ be the set of the nonnegative integers. For $s \in \mathbb{N}_0$, $H^s(I)$ will be the Sobolev space of functions having generalized derivatives up to order s in the space $L^2(I)$, and by $\|\cdot\|_{s,I}$ its usual norm, i.e. $\|g\|_{s,I} := \left(\sum_{\ell=0}^{s} \|\partial^\ell g\|_{0,I}^2\right)^{\frac{1}{2}}$ for $g \in H^s(I)$. Also, by $H_0^1(I)$ we denote the subspace of $H^1(I)$ consisting of functions which vanish at the endpoints of I in the sense of trace. We note that in $H_0^1(I)$ the, well-known, Poincaré-Friedrich inequality holds, i.e., there exists a nonegative constant C_{PF} such that

$$\|g\|_{0,I} \leq C_{PF}\, \|\partial g\|_{0,I} \quad \forall\, g \in H_0^1(I). \tag{2.1}$$

The sequence of pairs $\left((\lambda_k^2, \varepsilon_k)\right)_{k=1}^{\infty}$ is a solution to the eigenvalue/eigenfunction problem: find nonzero $\varphi \in H^2(D) \cap H_0^1(D)$ and $\sigma \in \mathbb{R}$ such that $-\partial^2 \varphi = \sigma\, \varphi$ in D. Since $(\varepsilon_k)_{k=1}^{\infty}$ is a complete $(\cdot,\cdot)_{0,D}-$orthonormal system in $L^2(D)$, for $s \in \mathbb{R}$, a subspace $\mathcal{V}^s(D)$ of $L^2(D)$ is defined by

$$\mathcal{V}^s(D) := \left\{ v \in L^2(D) : \quad \sum_{k=1}^{\infty} \lambda_k^{2s}\, (v, \varepsilon_k)_{0,D}^2 < \infty \right\}$$

which is provided with the norm $\|v\|_{\mathcal{V}^s} := \left(\sum_{k=1}^{\infty} \lambda_k^{2s}\, (v, \varepsilon_k)_{0,D}^2 \right)^{\frac{1}{2}} \quad \forall\, v \in \mathcal{V}^s(D)$. For $s \geq 0$, the pair $(\mathcal{V}^s(D), \|\cdot\|_{\mathcal{V}^s})$ is a complete subspace of $L^2(D)$ and we set $(\dot{\mathbf{H}}^s(D), \|\cdot\|_{\dot{\mathbf{H}}^s}) := (\mathcal{V}^s(D), \|\cdot\|_{\mathcal{V}^s})$. For $s < 0$, we define $(\dot{\mathbf{H}}^s(D), \|\cdot\|_{\dot{\mathbf{H}}^s})$ as the completion of $(\mathcal{V}^s(D), \|\cdot\|_{\mathcal{V}^s})$, or, equivalently, as the dual of $(\dot{\mathbf{H}}^{-s}(D), \|\cdot\|_{\dot{\mathbf{H}}^{-s}})$. Let $m \in \mathbb{N}_0$. It is well-known (see [21]) that

$$\dot{\mathbf{H}}^m(D) = \left\{ v \in H^m(D) : \quad \partial^{2i} v\,|_{\partial D} = 0 \quad \text{if} \quad 0 \leq i < \tfrac{m}{2} \right\} \tag{2.2}$$

and there exist positive constants $C_{m,A}$ and $C_{m,B}$ such that

$$C_{m,A}\, \|v\|_{m,D} \leq \|v\|_{\dot{\mathbf{H}}^m} \leq C_{m,B}\, \|v\|_{m,D}, \quad \forall\, v \in \dot{\mathbf{H}}^m(D). \tag{2.3}$$

Also, we define on $L^2(D)$ the negative norm $\|\cdot\|_{-m,D}$ by

$$\|v\|_{-m,D} := \sup\left\{ \tfrac{(v,\varphi)_{0,D}}{\|\varphi\|_{m,D}} : \quad \varphi \in \dot{\mathbf{H}}^m(D) \ \text{ and } \ \varphi \neq 0 \right\}, \quad \forall\, v \in L^2(D),$$

for which, using (2.3), it is easy to conclude that there exists a constant $C_{-m} > 0$ such that

$$\|v\|_{-m,D} \leq C_{-m}\, \|v\|_{\dot{\mathbf{H}}^{-m}}, \quad \forall\, v \in L^2(D). \tag{2.4}$$

Let $\mathbb{L}_2 = (L^2(D), (\cdot,\cdot)_{0,D})$ and $\mathcal{L}(\mathbb{L}_2)$ be the space of linear, bounded operators from $\mathbb{L}_2$ to $\mathbb{L}_2$. We say that, an operator $\Gamma \in \mathcal{L}(\mathbb{L}_2)$ is *Hilbert-Schmidt*, when $\|\Gamma\|_{\mathrm{HS}} := \left(\sum_{k=1}^{\infty} \|\Gamma(\varepsilon_k)\|_{0,D}^2\right)^{\frac{1}{2}} < +\infty$, where $\|\Gamma\|_{\mathrm{HS}}$ is the so called Hilbert-Schmidt norm of Γ. We note that the quantity $\|\Gamma\|_{\mathrm{HS}}$ does not change when we replace $(\varepsilon_k)_{k=1}^{\infty}$ by another complete orthonormal system of $\mathbb{L}_2$, as it is the sequence $(\varphi_k)_{k=0}^{\infty}$ with $\varphi_0(z) := 1$ and $\varphi_k(x) := \sqrt{2}\, \cos(\lambda_k\, z)$ for $k \in \mathbb{N}$ and $z \in \overline{D}$. It is well known (see, e.g., [7]) that an operator $\Gamma \in \mathcal{L}(\mathbb{L}_2)$ is Hilbert-Schmidt iff there exists a measurable function $g : D \times D \to \mathbb{R}$ such that $(\Gamma(v))(\cdot) = \int_D g(\cdot, y)\, v(y)\, dy$ for $v \in L^2(D)$, and then, it holds that

$$\|\Gamma\|_{\mathrm{HS}} = \left(\int_D \int_D g^2(x,y)\, dxdy \right)^{\frac{1}{2}}. \tag{2.5}$$

Let $\mathcal{L}_{\rm HS}(\mathbb{L}_2)$ be the set of Hilbert Schmidt operators of $\mathcal{L}(\mathbb{L}^2)$ and $\Phi : [0,T] \to \mathcal{L}_{\rm HS}(\mathbb{L}_2)$. Also, for a random variable X, let $\mathbb{E}[X]$ be its expected value, i.e., $\mathbb{E}[X] := \int_\Omega X\,dP$. Then, the Itô isometry property for stochastic integrals, which we will use often in the paper, reads

$$\mathbb{E}\left[\left\|\int_0^T \Phi\,dW\right\|_{0,D}^2\right] = \int_0^T \|\Phi(t)\|_{\rm HS}^2\,dt. \tag{2.6}$$

Let $\widehat{\Pi} : L^2((0,T)\times D) \to L^2((0,T)\times D)$ be a projection operator defined by

$$\widehat{\Pi} g(t,x) := \tfrac{1}{\Delta t}\sum_{i=1}^{J_\star+1}\left(\sum_{\ell=1}^{J_\star+1} G_{i,\ell}^{-1}\int_{T_n}\int_D g(s,y)\,\psi_\ell(y)\,dsdy\right)\psi_i(x), \quad \forall\,(t,x)\in T_n\times D, \tag{2.7}$$

for $n = 1,\dots,N_\star$ and for $g \in L^2((0,T)\times D)$, for which holds that

$$\left(\int_0^T\int_D (\widehat{\Pi} g)^2\,dxdt\right)^{\frac12} \le \left(\int_0^T\int_D g^2\,dxdt\right)^{\frac12}, \quad \forall\, g\in L^2((0,T)\times D). \tag{2.8}$$

Now, in the lemma below, we relate the stochastic integral of the projection $\widehat{\Pi}$ of a deterministic function to its space-time L^2-inner product with the discrete space-time white noise kernel $\widehat{W}$ defined in Section 1.1 (cf. Lemma 2.1 in [14]).

Lemma 2.1. *For $g \in L^2((0,T)\times D)$, it holds that*

$$\int_0^T\int_D \widehat{\Pi} g(t,x)\,dW(t,x) = \int_0^T\int_D \widehat{W}(s,y)\,g(s,y)\,dsdy. \tag{2.9}$$

Proof. To obtain (2.9) we work, using (2.7) and the properties of the stochastic integral, as follows:

$$\begin{aligned}
\int_0^T\int_D \widehat{\Pi} g(t,x)\,dW(t,x) =& \tfrac{1}{\Delta t}\sum_{n=1}^{N_\star}\sum_{i=1}^{J_\star+1}\sum_{\ell=1}^{J_\star+1} G_{i,\ell}^{-1}\left(\int_{T_n\times D} g(s,y)\,\psi_\ell(y)\,dsdy\right) R_{n,i}\\
=& \tfrac{1}{\Delta t}\sum_{n=1}^{N_\star}\int_{T_n\times D} g(s,y)\left(\sum_{i=1}^{J_\star+1}\sum_{\ell=1}^{J_\star+1} G_{i,\ell}^{-1}\psi_\ell(y)\,R_{n,i}\right)dsdy\\
=& \tfrac{1}{\Delta t}\sum_{n=1}^{N_\star}\int_0^T\int_D \mathcal{X}_{T_n}(s)\,g(s,y)\left(\sum_{i=1}^{J_\star+1}\sum_{\ell=1}^{J_\star+1} G_{\ell,i}^{-1}\,R_{n,i}\,\psi_\ell(y)\right)dsdy\\
=& \int_0^T\int_D g(s,y)\,\widehat{W}(s,y)\,dsdy.
\end{aligned}$$

□

We close this section by observing that: if $c_\star > 0$, then

$$\sum_{k=1}^{\infty}\lambda_k^{-(1+c_\star\epsilon)} \le \left(\tfrac{1+2c_\star}{c_\star\pi}\right)\tfrac{1}{\epsilon}, \quad \forall\,\epsilon\in(0,2], \tag{2.10}$$

and if $(\mathcal{H},(\cdot,\cdot)_{\mathcal{H}})$ is a real inner product space, then

$$(g-v,g)_{\mathcal{H}} \ge \tfrac12\left[\,(g,g)_{\mathcal{H}} - (v,v)_{\mathcal{H}}\,\right], \quad \forall\, g,v\in\mathcal{H}. \tag{2.11}$$

2.2. **Linear elliptic and parabolic operators.** Let us define the elliptic differential operators Λ_B, $\widetilde{\Lambda}_B$: $\dot{\mathbf{H}}^4(D) \to L^2(D)$ by $\Lambda_B v := \partial^4 v + \mu\,\partial^2 v$ and $\widetilde{\Lambda}_B v := \Lambda_B v + \mu^2\,v$ for $v \in \dot{\mathbf{H}}^4(D)$, and consider the corresponding Dirichlet fourth-order two-point boundary value problems: given $f \in L^2(D)$ find v_B, $\widetilde{v}_B \in \dot{\mathbf{H}}^4(D)$ such that

$$\Lambda_B v_B = f \quad \text{in} \quad D \tag{2.12}$$

and

$$\widetilde{\Lambda}_B \widetilde{v}_B = f \quad \text{in} \quad D. \tag{2.13}$$

Assumption (1.2) yields that when $\kappa = 1$ or $\kappa \geq 2$ and $\mu \neq \lambda_{\kappa-1}^2$, the operator Λ_B is invertible and thus the problem (2.12) is well-posed. However, the problem (2.13) is always well-posed. Letting T_B, $\widetilde{T}_B$: $L^2(D) \to \dot{\mathbf{H}}^4(D)$ be the solution operator of (2.12) and (2.13), respectively, i.e. $T_B f := \Lambda_B^{-1} f = v_B$ and $\widetilde{T}_B f := \widetilde{\Lambda}_B^{-1} f = \widetilde{v}_B$, it is easy to verify that

$$T_B f = \sum_{k=1}^{\infty} \tfrac{(\varepsilon_k, f)_{0,D}}{\lambda_k^2(\lambda_k^2-\mu)}\ \varepsilon_k \quad \text{and} \quad \widetilde{T}_B f = \sum_{k=1}^{\infty} \tfrac{(\varepsilon_k, f)_{0,D}}{\lambda_k^2(\lambda_k^2-\mu)+\mu^2}\ \varepsilon_k, \quad \forall\, f \in L^2(D), \tag{2.14}$$

and

$$\|T_B f\|_{m,D} + \|\widetilde{T}_B f\|_{m,D} \leq C_{R,m}\, \|f\|_{m-4,D}, \quad \forall\, f \in H^{\max\{0,m-4\}}(D), \quad \forall\, m \in \mathbb{N}_0, \tag{2.15}$$

where $C_{R,m}$ is a positive constant which is independent of f but depends on the D and m. Observing that

$$(\widetilde{T}_B v_1, v_2)_{0,D} = (v_1, \widetilde{T}_B v_2)_{0,D}, \quad \forall\, v_1, v_2 \in L^2(D),$$

and in view (2.14), the map $\widetilde{\gamma}_B : L^2(D) \times L^2(D) \to \mathbb{R}$ defined by

$$\widetilde{\gamma}_B(v, w) = (\widetilde{T}_B v, w)_{0,D} \quad \forall\, v, w \in L^2(D),$$

is an inner product on $L^2(D)$.

Let $(\mathcal{S}(t)w_0)_{t\in[0,T]}$ be the standard semigroup notation for the solution w of (1.5). Then, the following a priori bounds hold (see Appendix A): for $\ell \in \mathbb{N}_0$, $\beta \geq 0$ and $p \geq 0$, there exists a constant $\mathcal{C}_{\beta,\ell,\mu,\mu T} > 0$ such that:

$$\int_{t_a}^{t_b} (\tau - t_a)^{\beta} \left\|\partial_t^{\ell} \mathcal{S}(\tau) w_0\right\|_{\dot{\mathbf{H}}^p}^2 d\tau \leq \mathcal{C}_{\beta,\ell,\mu,\mu T}\, \|w_0\|_{\dot{\mathbf{H}}^{p+4\ell-2\beta-2}}^2 \tag{2.16}$$

forall $w_0 \in \dot{\mathbf{H}}^{p+4\ell-2\beta-2}(D)$ and t_a, $t_b \in [0, T]$ with $t_b > t_a$.

2.3. **Discrete spaces and operators.** For $r \in \{2, 3\}$, let $M_h^r \subset H_0^1(D) \cap H^2(D)$ be a finite element space consisting of functions which are piecewise polynomials of degree at most r over a partition of D in intervals with maximum mesh-length h. It is well-known (cf., e.g., [5]) that the following approximation property holds:

$$\inf_{\chi \in M_h^r} \|v - \chi\|_{2,D} \leq C_{FM,r}\, h^{s-1}\, \|v\|_{s+1,D}, \quad \forall\, v \in H^{s+1}(D) \cap H_0^1(D), \quad \forall\, s \in \{2, r\}, \tag{2.17}$$

where $C_{FM,r}$ is a positive constant that depends on r and is independent of h and v. Then, we define the discrete elliptic operators $\Lambda_{B,h}$, $\widetilde{\Lambda}_{B,h} : M_h^r \to M_h^r$ by

$$(\Lambda_{B,h}\varphi, \chi)_{0,D} := (\partial^2\varphi, \partial^2\chi)_{0,D} + \mu\, (\partial^2\varphi, \chi)_{0,D}, \quad \forall\, \varphi, \chi \in M_h^r, \tag{2.18}$$

and

$$\widetilde{\Lambda}_{B,h}\varphi := \Lambda_{B,h}\varphi + \mu^2\, \varphi, \quad \forall\, \varphi \in M_h^r. \tag{2.19}$$

Also, let $P_h : L^2(D) \to M_h^r$ be the usual $L^2(D)-$projection operator onto M_h^r for which it holds that

$$(P_h f, \chi)_{0,D} = (f, \chi)_{0,D}, \quad \forall\, \chi \in M_h^r, \quad \forall\, f \in L^2(D).$$

A finite element approximation $\widetilde{v}_{B,h} \in M_h^r$ of the solution $\widetilde{v}_B$ of (2.13) is defined by the requirement

$$\widetilde{\Lambda}_{B,h}\widetilde{v}_{B,h} = P_h f, \tag{2.20}$$

where the operator $\widetilde{\Lambda}_{B,h}$ is invertible since

$$(\widetilde{\Lambda}_{B,h}\chi, \chi)_{0,D} \geq \tfrac{1}{2}\, \left(\|\partial^2\chi\|_{0,D}^2 + \mu^2\, \|\chi\|_{0,D}^2 \right), \quad \forall \chi \in M_h^r. \tag{2.21}$$

Thus, we denote by $\widetilde{T}_{B,h} : L^2(D) \to M_h^r$ the solution operator of (2.20), i.e.

$$\widetilde{T}_{B,h} f := \widetilde{v}_{B,h} = \widetilde{\Lambda}_{B,h}^{-1} P_h f, \quad \forall\, f \in L^2(D).$$

Next, we derive an $L^2(D)$ error estimate for the finite element method (2.20).

Proposition 2.1. *Let $r \in \{2,3\}$. Then we have*

$$\|\widetilde{T}_B f - \widetilde{T}_{B,h} f\|_{0,D} \leq C \begin{cases} h^4 \, \|f\|_{0,D}, & r=3, \\ h^3 \, \|f\|_{-1,D}, & r=3, \\ h^2 \, \|f\|_{-1,D}, & r=2, \end{cases} \qquad \forall\, f \in L^2(D), \tag{2.22}$$

where C is a positive constant independent of h and f.

Proof. Let $f \in L^2(D)$, $e = \widetilde{T}_B f - \widetilde{T}_{B,h} f$ and $\widetilde{v} = \widetilde{T}_B e$. To simplify the notation we define $\mathcal{B} : H^2(D) \times H^2(D) \to \mathbb{R}$ by $\mathcal{B}(v,w) := (\partial^2 v, \partial^2 w)_{0,D} + \mu\, (\partial^2 v, w)_{0,D} + \mu^2\, (v,w)_{0,D}$ for v, $w \in H^2(D)$. It is easily seen that

$$\begin{aligned} \mathcal{B}(v,w) &\leq \sqrt{2}\,(1+\mu)\, \left(\|\partial^2 v\|^2_{0,D} + \mu^2\, \|v\|^2_{0,D} \right)^{\frac{1}{2}} \|w\|_{2,D} \quad \forall\, v, w \in H^2(D), \\ \mathcal{B}(v,v) &\geq \tfrac{1}{2} \left[\|\partial^2 v\|^2_{0,D} + \mu^2\, \|v\|^2_{0,D} \right] \quad \forall\, v \in H^2(D). \end{aligned} \tag{2.23}$$

Later in the proof we shall use the symbol C for a generic constant that is independent of h and f, and may changes value from one line to the other.

First, we observe that $\|e\|^2_{0,D} = \mathcal{B}(e, \widetilde{v})$. Then, we use the Galerkin orthogonality to get

$$\|e\|^2_{0,D} = \mathcal{B}(e, \widetilde{v} - \chi), \quad \forall\, \chi \in M_h^r,$$

which, along with (2.23), leads to

$$\|e\|^2_{0,D} \leq C \left(\|\partial^2 e\|^2_{0,D} + \mu^2\, \|e\|^2_{0,D} \right)^{\frac{1}{2}} \inf_{\chi \in M_h^r} \|\widetilde{v} - \chi\|_{2,D}. \tag{2.24}$$

Using again (2.23) and the Galerkin orthogonality, we obtain

$$\begin{aligned} \|\partial^2 e\|^2_{0,D} + \mu^2\, \|e\|^2_{0,D} &\leq 2\, \mathcal{B}(e,e) \\ &\leq 2\, \mathcal{B}(e, \widetilde{T}_B f - \chi) \\ &\leq C \left(\|\partial^2 e\|^2_{0,D} + \mu^2\, \|e\|^2_{0,D} \right)^{\frac{1}{2}} \|\widetilde{T}_B f - \chi\|_{2,D}, \quad \forall\, \chi \in M_h^r, \end{aligned}$$

which yields that

$$\left(\|\partial^2 e\|^2_{0,D} + \mu^2\, \|e\|^2_{0,D} \right)^{\frac{1}{2}} \leq C \inf_{\chi \in M_h^r} \|\widetilde{T}_B f - \chi\|_{2,D}. \tag{2.25}$$

Combining (2.24), (2.25) and (2.17), we arrive at

$$\begin{aligned} \|e\|^2_{0,D} &\leq C \inf_{\chi \in M_h^r} \|\widetilde{T}_B f - \chi\|_{2,D} \inf_{\chi \in M_h^r} \|\widetilde{v} - \chi\|_{2,D} \\ &\leq C\, h^{s+s'-2}\, \|\widetilde{T}_B f\|_{s+1,D}\, \|\widetilde{T}_B e\|_{s'+1,D}, \quad \forall\, s, s' \in \{2, r\}. \end{aligned} \tag{2.26}$$

Let $r=2$. We use (2.26) and (2.15) to get

$$\begin{aligned} \|e\|^2_{0,D} &\leq C\, h^2\, \|\widetilde{T}_B f\|_{3,D}\, \|\widetilde{T}_B e\|_{3,D} \\ &\leq C\, h^2\, \|f\|_{-1,D}\, \|e\|_{-1,D} \\ &\leq C\, h^2\, \|f\|_{-1,D}\, \|e\|_{0,D}, \end{aligned}$$

from which we conclude (2.22) for $r=2$.

Let $r=3$. We use (2.26) with $s'=3$ and (2.15) to obtain

$$\begin{aligned} \|e\|^2_{0,D} &\leq C\, h^{s+1}\, \|\widetilde{T}_B f\|_{s+1,D}\, \|\widetilde{T}_B e\|_{4,D} \\ &\leq C\, h^{s+1}\, \|f\|_{s-3,D}\, \|e\|_{0,D}, \quad s = 2, 3, \end{aligned}$$

from which we conclude (2.22) for $r=3$. $\square$

Let $\widetilde{\gamma}_{B,h} : L^2(D) \times L^2(D) \to \mathbb{R}$ be defined by

$$\widetilde{\gamma}_{B,h}(f,g) = (\widetilde{T}_{B,h} f, g)_{0,D} \quad \forall\, f, g \in L^2(D).$$

Then, as a simple consequence of (2.21), the following inequality holds

$$\widetilde{\gamma}_{B,h}(f,f) \geq \tfrac{1}{2} \left(\|\partial^2(\widetilde{T}_{B,h} f)\|^2_{0,D} + \mu^2 \, \|\widetilde{T}_{B,h} f\|^2_{0,D} \right), \quad \forall\, f \in L^2(D). \tag{2.27}$$

Thus, observing that

$$(\widetilde{T}_{B,h} f, g)_{0,D} = (f, \widetilde{T}_{B,h} g)_{0,D}, \quad \forall\, f, g \in L^2(D),$$

and using (2.27), we easily conclude that $\widetilde{\gamma}_{B,h}$ is an inner product in $L^2(D)$. We close this section with the following useful lemma.

Lemma 2.2. *There exists a positive constant $C > 0$ such that*

$$\widetilde{\gamma}_{B,h}(f,f) \leq \, C \, \|f\|^2_{-2,D}, \quad \forall\, f \in L^2(D). \tag{2.28}$$

Proof. Let $f \in L^2(D)$, $\psi = \widetilde{T}_B f$ and $\psi_h = \widetilde{T}_{B,h} f$. Then, we have

$$\begin{aligned} (\widetilde{T}_{B,h} f, f)_{0,D} &= (\widetilde{\Lambda}_B \psi, \psi_h)_{0,D} \\ &= (\partial^2 \psi, \partial^2 \psi_h)_{0,D} + \mu \, (\partial^2 \psi, \psi_h)_{0,D} + \mu^2 \, (\psi, \psi_h)_{0,D} \\ &\leq \tfrac{1}{\varepsilon} \, \left(\|\partial^2 \psi\|^2_{0,D} + \mu^2 \, \|\psi\|^2_{0,D} \right) + \varepsilon \, \left(\|\partial^2 \psi_h\|^2_{0,D} + \mu^2 \, \|\psi_h\|^2_{0,D} \right), \quad \forall\, \varepsilon > 0. \end{aligned} \tag{2.29}$$

Setting $\varepsilon = \frac{1}{4}$ in (2.29) and then combining it with (2.27), we obtain

$$\|\partial^2 \psi_h\|^2_{0,D} + \mu^2 \, \|\psi_h\|^2_{0,D} \leq \, 16 \, \left(\|\partial^2 \psi\|^2_{0,D} + \mu^2 \, \|\psi\|^2_{0,D} \right). \tag{2.30}$$

Finally, (2.29) with $\varepsilon = \frac{1}{4}$, (2.30) and (2.15) yield

$$\begin{aligned} \widetilde{\gamma}_{B,h}(f,f) &\leq 8 \, \left(\|\partial^2 \psi\|^2_{0,D} + \mu^2 \, \|\psi\|^2_{0,D} \right) \\ &\leq 8 \, (1 + \mu^2) \, \|\widetilde{T}_B f\|^2_{2,D} \\ &\leq 8 \, (1 + \mu^2) \, C_{R,2} \, \|f\|^2_{-2,D}. \end{aligned}$$

Thus, we arrived at (2.28). □

3. An Estimate for the Modeling Error

In this section, we estimate the modeling error in terms of Δt and Δx (cf. Theorem 3.1 in [14]).

Theorem 3.1. *Let u be the solution of* (1.1) *and $\widehat{u}$ be the solution of* (1.6)*. Then, there exists a real constant $\widetilde{C} > 0$, independent of Δt and Δx, such that*

$$\max_{[0,T]} \left(\mathbb{E} \left[\|u - \widehat{u}\|^2_{0,D} \right] \right)^{\frac{1}{2}} \leq \, \widetilde{C} \, \left[\omega_0(\Delta t) \, \Delta t^{\frac{1}{8}} + \epsilon^{-\frac{1}{2}} \, \Delta x^{\frac{1}{2} - \epsilon} \right], \quad \forall\, \epsilon \in \left(0, \tfrac{1}{2}\right], \tag{3.1}$$

where $\omega_0(\Delta t) := \sqrt{1 + \Delta t^{\frac{3}{4}}}$.

Proof. Using (1.3), (1.7) and Lemma 2.1, we conclude that

$$u(t,x) - \widehat{u}(t,x) = \int_0^T \!\! \int_D \left[\mathcal{X}_{(0,t)}(s) \, \Psi(t-s; x, y) - \widetilde{\Psi}(t,x;s,y) \right] dW(s,y), \quad \forall\, (t,x) \in [0,T] \times \overline{D}, \tag{3.2}$$

where $\widetilde{\Psi} : (0,T) \times D \to L^2((0,T) \times D)$ is given by

$$\widetilde{\Psi}(t,x;s,y) := \tfrac{1}{\Delta t} \int_{T_n} \mathcal{X}_{(0,t)}(s') \, \left[\sum_{i=1}^{J_\star + 1} \psi_i(y) \left(\sum_{\ell=1}^{J_\star+1} G^{-1}_{i,\ell} \, \int_D \Psi(t - s'; x, y') \, \psi_\ell(y') \, dy' \right) \right] \, ds', \quad \forall\, (s,y) \in T_n \times D,$$

for $n = 1, \ldots, N_\star$.

Let $\Theta := \left\{ \mathbb{E} \left[\|u - \widehat{u}\|^2_{0,D} \right] \right\}^{\frac{1}{2}}$ and $t \in (0,T]$. Using (3.2) and Itô isometry (2.6), we obtain

$$\Theta(t) = \left\{ \int_0^T \int_D \!\! \int_D \left[\mathcal{X}_{(0,t)}(s) \, \Psi(t-s; x, y) - \widetilde{\Psi}(t,x;s,y) \right]^2 \, dxdyds \right\}^{\frac{1}{2}}.$$

Now, we introduce the splitting

$$\Theta(t) \leq \Theta_{A}(t) + \Theta_{B}(t), \tag{3.3}$$

where

$$\Theta_{A}(t) := \left\{ \sum_{n=1}^{N_\star} \int_D \int_D \int_{T_n} \left[\tfrac{1}{\Delta t} \int_{T_n} \mathcal{X}_{(0,t)}(s')\,\Psi(t-s';x,y)\,ds' - \widetilde{\Psi}(t,x;s,y) \right]^2 dxdyds \right\}^{\frac{1}{2}}$$

and

$$\Theta_{B}(t) := \left\{ \sum_{n=1}^{N_\star} \int_D \int_D \int_{T_n} \left[\mathcal{X}_{(0,t)}(s)\,\Psi(t-s;x,y) - \tfrac{1}{\Delta t} \int_{T_n} \mathcal{X}_{(0,t)}(s')\,\Psi(t-s';x,y)\,ds' \right]^2 dxdyds \right\}^{\frac{1}{2}}.$$

Also, to simplify the notation in the rest of the proof, we set $\mu_k := \lambda_k^2\,(\lambda_k^2-\mu)$ for $k\in\mathbb{N}$, and use the symbol C to denote a generic constant that is independent of Δt and Δx and may changes value from one line to the other.

- `Estimation of` $\Theta_{A}(t)$: Using (1.4) and the $(\cdot,\cdot)_{0,D}-$orthogonality of $(\varepsilon_k)_{k=1}^{\infty}$, we have

$$\begin{aligned}\Theta_{A}^2(t) &= \tfrac{1}{\Delta t} \sum_{n=1}^{N_\star} \int_D \int_D \left[\int_{T_n} \mathcal{X}_{(0,t)}(s') \left[\Psi(t-s';x,y) - \sum_{\ell,i=1}^{J_\star+1} G_{i,\ell}^{-1}\,\left(\Psi(t-s';x,\cdot),\psi_\ell(\cdot)\right)_{0,D}\,\psi_i(y) \right] ds' \right]^2 dydx \\ &= \tfrac{1}{\Delta t} \sum_{n=1}^{N_\star} \left[\sum_{k=1}^{\infty} \left(\int_{T_n} \mathcal{X}_{(0,t)}(s')\,e^{-\mu_k(t-s')}\,ds' \right)^2 \int_D \left(\varepsilon_k'(y) - \sum_{\ell,i=1}^{J_\star+1} G_{i,\ell}^{-1}\,(\varepsilon_k',\psi_\ell)_{0,D}\,\psi_i(y) \right)^2 dy \right]\end{aligned}$$

from which, using the Cauchy-Schwarz inequality, follows that

$$\Theta_{A}^2(t) \leq \sum_{k=1}^{\kappa} A_k(t)\,B_k + \sum_{k=\kappa+1}^{\infty} A_k(t)\,B_k, \tag{3.4}$$

where

$$\begin{aligned} A_k(t) &:= 2\,\lambda_k^2 \int_0^t e^{-2\mu_k(t-s')}\,ds', \\ B_k &:= \int_D \left(\varphi_k(y) - \sum_{\ell,i=1}^{J_\star+1} G_{i,\ell}^{-1}\,(\varphi_k,\psi_\ell)_{0,D}\,\psi_i(y) \right)^2 dy. \end{aligned}$$

First, we observe that

$$\begin{aligned} \sqrt{B_k} &\leq \max_{1\leq j\leq J_\star} \sup_{x,y\in D_j} |\,\varphi_k(x)-\varphi_k(y)\,| \\ &\leq \min\{1,\,\lambda_k\Delta x\} \\ &\leq \min\left\{1,(\sqrt{2}\lambda_k\Delta x)^\theta\right\}, \quad \forall\,\theta\in[0,1], \quad \forall\,k\in\mathbb{N}. \end{aligned} \tag{3.5}$$

Next, we use (1.2), to obtain

$$\begin{aligned} A_k(t) &\leq \tfrac{1-e^{-2\mu_k t}}{\lambda_k^2-\mu} \\ &< \tfrac{(\kappa+1)^2}{1+2\,\kappa}\,\tfrac{1}{\lambda_k^2}, \quad \forall\,k\geq\kappa+1. \end{aligned} \tag{3.6}$$

Thus, from (3.4), (3.5) and (3.6), we conclude that

$$\Theta_{A}^2(t) \leq C\,\left((\Delta x)^2 \sum_{k=1}^{\kappa} \lambda_k^2 + (\Delta x)^{2\theta} \sum_{k=\kappa+1}^{\infty} \tfrac{1}{\lambda_k^{2-2\theta}} \right)$$

which yields

$$\Theta_{A}(t) \leq C\,(\Delta x)^\theta \, \left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{1+2(\frac{1}{2}-\theta)}} \right)^{\frac{1}{2}}, \quad \forall\,\theta\in[0,\tfrac{1}{2}). \tag{3.7}$$

- `Estimation of` $\Theta_{B}(t)$: For $t\in(0,T]$, let $\widehat{N}(t):=\min\left\{\ell\in\mathbb{N}:\quad 1\le\ell\le N_\star \ \text{ and }\ t\le t_\ell\right\}$ and

$$\widehat{T}_n(t):=T_n\cap(0,t)=\begin{cases}T_n, & \text{if } \ n<\widehat{N}(t)\\ (t_{\widehat{N}(t)-1},t), & \text{if } \ n=\widehat{N}(t)\end{cases},\qquad n=1,\dots,\widehat{N}(t).$$

Thus, using (1.4) and the $(\cdot,\cdot)_{0,D}-$orthogonality of $(\varepsilon_k)_{k=1}^\infty$ and $(\varphi_k)_{k=1}^\infty$ as follows

$$\begin{aligned}\Theta_B^2(t)&=\tfrac{1}{(\Delta t)^2}\sum_{n=1}^{N_\star}\int_D\int_D\int_{T_n}\left[\int_{T_n}\left[\mathcal{X}_{(0,t)}(s)\,\Psi(t-s;x,y)-\mathcal{X}_{(0,t)}(s')\,\Psi(t-s';x,y)\right]ds'\right]^2dxdyds\\ &=\tfrac{1}{(\Delta t)^2}\sum_{n=1}^{N_\star}\int_D\int_D\int_{T_n}\left[\sum_{k=1}^\infty\lambda_k\,\varepsilon_k(x)\,\varphi_k(y)\int_{T_n}\left[\mathcal{X}_{(0,t)}(s)\,e^{-\mu_k(t-s)}-\mathcal{X}_{(0,t)}(s')\,e^{-\mu_k(t-s')}\right]ds'\right]^2dxdyds\end{aligned}$$

we conclude that

$$\Theta_B^2(t)\le\sum_{k=1}^\infty\lambda_k^2\left(\tfrac{1}{(\Delta t)^2}\sum_{n=1}^{\widehat{N}(t)}\Psi_n^k(t)\right),\tag{3.8}$$

where

$$\Psi_n^k(t):=\int_{T_n}\left[\int_{T_n}\left(\mathcal{X}_{(0,t)}(s)\,e^{-\mu_k(t-s)}-\mathcal{X}_{(0,t)}(s')\,e^{-\mu_k(t-s')}\right)\,ds'\right]^2ds.$$

Let $k\in\mathbb{N}$ and $n\in\{1,\dots,\widehat{N}(t)-1\}$. Then, we have

$$\begin{aligned}\Psi_n^k(t)&=\int_{T_n}\Big(\int_{T_n}\int_s^{s'}\mu_k\,e^{-\mu_k(t-\tau)}\,d\tau ds'\Big)^2ds\\ &\le\int_{T_n}\Big(\int_{T_n}\int_{t_{n-1}}^{\max\{s',s\}}\mu_k\,e^{-\mu_k(t-\tau)}\,d\tau ds'\Big)^2ds\\ &\le 2\int_{T_n}\Big(\int_{T_n}\int_{t_{n-1}}^{s'}\mu_k\,e^{-\mu_k(t-\tau)}\,d\tau ds'\Big)^2ds+2\int_{T_n}\Big(\int_{T_n}\int_{t_{n-1}}^{s}\mu_k\,e^{-\mu_k(t-\tau)}\,d\tau\,ds'\Big)^2ds\\ &\le 2\,\Delta t\Big(\int_{T_n}\int_{t_{n-1}}^{s'}\mu_k\,e^{-\mu_k(t-\tau)}\,d\tau ds'\Big)^2+2\,(\Delta t)^2\int_{T_n}\Big(\int_{t_{n-1}}^{s}\mu_k\,e^{-\mu_k(t-\tau)}\,d\tau\Big)^2ds,\end{aligned}$$

from which, after using the Cauchy-Schwarz inequality, we arrive at

$$\Psi_n^k(t)\le 4\,(\Delta t)^2\int_{T_n}\Big(\int_{t_{n-1}}^{s}\mu_k\,e^{-\mu_k(t-\tau)}\,d\tau\Big)^2ds.\tag{3.9}$$

For $k\le\kappa$, we use (3.9) to get

$$\Psi_n^k(t)\le 4\max_{1\le k\le\kappa}(\mu_k)^2\,(\Delta t)^5.\tag{3.10}$$

For $k\ge\kappa+1$, we use (3.9) to have

$$\begin{aligned}\Psi_n^k(t)&\le 4\,(\Delta t)^2\int_{T_n}\Big(e^{-\mu_k(t-s)}-e^{-\mu_k(t-t_{n-1})}\Big)^2ds\\ &\le 4\,(\Delta t)^2\left(1-e^{-\mu_k\Delta t}\right)^2\int_{T_n}e^{-2\mu_k(t-s)}\,ds\\ &\le 2\,(\Delta t)^2\left(1-e^{-\mu_k\Delta t}\right)^2\tfrac{e^{-\mu_k(t-t_n)}-e^{-\mu_k(t-t_{n-1})}}{\mu_k}.\end{aligned}\tag{3.11}$$

Summing with respect to n, and using (3.9), (3.10) and (3.11), we obtain

$$\tfrac{1}{(\Delta t)^2}\sum_{n=1}^{\widehat{N}(t)-1}\Psi_n^k(t)\le C\begin{cases}(\Delta t)^2, & k\le\kappa,\\ \frac{(1-e^{-\mu_k\Delta t})^2}{\mu_k}, & k\ge\kappa+1\end{cases}.\tag{3.12}$$

Considering, now, the case $n = \widehat{N}(t)$, we have

$$\Psi^k_{\widehat{N}(t)}(t) = \Psi^k_{A}(t) + \Psi^k_{B}(t) \tag{3.13}$$

with

$$\begin{aligned}
\Psi^k_{A}(t) &:= \int_{t_{\widehat{N}(t)-1}}^{t} \left(\int_{t_{\widehat{N}(t)-1}}^{t} \int_{s'}^{s} \mu_k \, e^{-\mu_k (t-\tau)} \, d\tau ds' + \int_{t}^{t_{\widehat{N}(t)}} e^{-\mu_k (t-s)} \, ds' \right)^2 ds, \\
\Psi^k_{B}(t) &:= \int_{t}^{t_{\widehat{N}(t)}} \left(\int_{t_{\widehat{N}(t)}-1}^{t} e^{-\mu_k (t-s')} \, ds' \right)^2 ds.
\end{aligned}$$

For $k \le \kappa$, we obtain

$$\tfrac{1}{(\Delta t)^2} \, \Psi^k_{\widehat{N}(t)}(t) \le \, C \, \Delta t. \tag{3.14}$$

For $k \ge \kappa + 1$, we have

$$\begin{aligned}
\Psi^k_{B}(t) &\le \tfrac{\Delta t}{\mu_k^2} \left[1 - e^{-\mu_k \left(t - t_{\widehat{N}(t)-1} \right)} \right]^2 \\
&\le \tfrac{\Delta t}{\mu_k^2} \left(1 - e^{-\mu_k \, \Delta t} \right)^2
\end{aligned}$$

and

$$\begin{aligned}
\Psi^k_{A}(t) &\le \int_{t_{\widehat{N}(t)-1}}^{t} \left[\int_{t_{\widehat{N}(t)-1}}^{t} \int_{s'}^{s} \mu_k \, e^{-\mu_k (t-\tau)} \, d\tau ds' + \Delta t \;\; e^{-\mu_k (t-s)} \right]^2 ds \\
&\le 2 \int_{t_{\widehat{N}(t)-1}}^{t} \left[\int_{t_{\widehat{N}(t)-1}}^{t} \int_{s'}^{s} \mu_k \, e^{-\mu_k (t-\tau)} \, d\tau ds' \right]^2 ds + \tfrac{(\Delta t)^2}{\mu_k} \left[1 - e^{-2\mu_k \left(t - t_{\widehat{N}(t)-1} \right)} \right] \\
&\le 2 \int_{t_{\widehat{N}(t)-1}}^{t} \left[\int_{t_{\widehat{N}(t)-1}}^{t} \int_{t_{\widehat{N}(t)-1}}^{\max\{s,s'\}} \mu_k \, e^{-\mu_k (t-\tau)} \, d\tau ds' \right]^2 ds + \tfrac{(\Delta t)^2}{\mu_k} \left(1 - e^{-2\mu_k \, \Delta t} \right) \\
&\le 8 \, (\Delta t)^2 \int_{t_{\widehat{N}(t)-1}}^{t} \left[\int_{t_{\widehat{N}(t)-1}}^{s} \mu_k \, e^{-\mu_k (t-\tau)} \, d\tau \right]^2 ds + \tfrac{(\Delta t)^2}{\mu_k} \left(1 - e^{-2\mu_k \, \Delta t} \right) \\
&\le 8 \, (\Delta t)^2 \int_{t_{\widehat{N}(t)-1}}^{t} \left[e^{-\mu_k (t-s)} - e^{-\mu_k (t - t_{\widehat{N}(t)-1})} \right]^2 ds + \tfrac{(\Delta t)^2}{\mu_k} \left(1 - e^{-2\mu_k \, \Delta t} \right),
\end{aligned}$$

which, along with (3.13), gives

$$\Psi^k_{\widehat{N}(t)} \le 5 \tfrac{(\Delta t)^2}{\mu_k} \left(1 - e^{-2\mu_k \, \Delta t} \right) + \tfrac{\Delta t}{\mu_k^2} \left(1 - e^{-\mu_k \Delta t} \right)^2 .$$

Since the mean value theorem yields: $1 - e^{-\mu_k \Delta t} \le \mu_k \, \Delta t$, the above inequality takes the form

$$\tfrac{1}{(\Delta t)^2} \, \Psi^k_{\widehat{N}(t)} \le 6 \, \tfrac{1 - e^{-2\mu_k \, \Delta t}}{\mu_k} \, . \tag{3.15}$$

Combining (3.8), (3.12), (3.14) and (3.15) we obtain

$$\begin{aligned}
\Theta^2_{B}(t) &\le C \left[\Delta t + \sum_{k=\kappa+1}^{\infty} \lambda_k^2 \, \tfrac{1 - e^{-2 \, \mu_k \, \Delta t}}{\mu_k} \right] \\
&\le C \left[\Delta t + \sum_{k=1}^{\infty} \tfrac{1 - e^{-c_0 \, \lambda_k^4 \, \Delta t}}{\lambda_k^2} \right],
\end{aligned} \tag{3.16}$$

with $c_0=\frac{2\,(1+2\kappa)}{(\kappa+1)^2}$. To get a convergence estimate we have to exploit the way the series depends on Δt in the above relation:

$$
\begin{aligned}
\sum_{k=1}^{\infty}\tfrac{1-e^{-c_0\,\lambda_k^4\,\Delta t}}{\lambda_k^2}&\le\tfrac{1-e^{-c_0\,\pi^4\,\Delta t}}{\pi^2}+\int_1^{\infty}\tfrac{1-e^{-c_0\,x^4\,\pi^4\,\Delta t}}{x^2\,\pi^2}\,dx\\
&\le C\,\left[\left(1-e^{-c_0\,\pi^4\,\Delta t}\right)+\Delta t\int_1^{\infty}x^2\,e^{-c_0\,x^4\,\pi^4\,\Delta t}\,dx\right]\\
&\le C\,\left[\Delta t+(\Delta t)^{\frac14}\int_0^{\infty}y^2\,e^{-2y^4}\,dy\right]\\
&\le C\,\left[(\Delta t)^{\frac34}+1\right]\,(\Delta t)^{\frac14}.
\end{aligned}\tag{3.17}
$$

Using the bounds (3.16) and (3.17) we conclude that

$$\Theta_B(t)\le\,C\,\left[(\Delta t)^{\frac34}+1\right]^{\frac12}\,\Delta t^{\frac18}.\tag{3.18}$$

The error bound (3.1) follows by observing that $\Theta(0)=0$ and combining the bounds (3.3), (3.7), (3.18) and (2.10). □

4. Time-Discrete Approximations

The Backward Euler time-stepping method for problem (1.6) specifies an approximation $\widehat{U}^m$ of $\widehat{u}(\tau_m,\cdot)$ starting by setting

$$\widehat{U}^0:=0,\tag{4.1}$$

and then, for $m=1,\dots,M$, by finding $\widehat{U}^m\in\dot{\mathbf{H}}^4(D)$ such that

$$\widehat{U}^m-\widehat{U}^{m-1}+\Delta\tau\,\Lambda_B\widehat{U}^m=\int_{\Delta_m}\partial_x\widehat{W}\,ds\quad\text{a.s..}\tag{4.2}$$

The method is well-defined when the differential operator $Q_{B,\Delta\tau}:=I+\Delta\tau\,\Lambda_B:\dot{\mathbf{H}}^4(D)\to L^2(D)$ is invertible. It is easily seen that $Q_{B,\Delta\tau}$ is invertible when $1+\Delta\tau\,\lambda_k^2\,(\lambda_k^2-\mu)\neq0$ for $k\in\mathbb{N}$, or equivalently when: $\kappa=1$ or $\kappa\ge2$ and $\Delta\tau\max\limits_{1\le k\le\kappa-1}\lambda_k^2\,(\mu-\lambda_k^2)\neq1$. If $\kappa\ge2$, then it is easily seen that $\max\limits_{1\le k\le\kappa-1}\lambda_k^2\,(\mu-\lambda_k^2)\le\frac{\mu^2}{4}$, so the condition $\Delta\tau\,\frac{\mu^2}{4}<1$ is a sufficient condition for the invertibility of $Q_{B,\Delta\tau}$.

4.1. The Deterministic Case.

The Backward Euler time-discrete approximations of the solution w to the deterministic problem (1.5) are defined as follows: first we set

$$W^0:=w_0,\tag{4.3}$$

and then, for $m=1,\dots,M$, we find $W^m\in\dot{\mathbf{H}}^4(D)$ such that

$$W^m-W^{m-1}+\Delta\tau\,\Lambda_BW^m=0.\tag{4.4}$$

Obviously, the Backward Euler time-discrete approximations are well-defined when $Q_{B,\Delta\tau}$ is invertible. Our next step, is to derive an error estimate in a discrete in time $L_t^2(L_x^2)$ norm, taking into account that, in constrast to the case $\mu=0$ considered in [14], the operator Λ_B is not always invertible.

Proposition 4.1. *Let $(W^m)_{m=0}^M$ be the Backward Euler time-discrete approximations of the solution w of the problem (1.5) defined in (4.3)–(4.4). Also, we assume that $\kappa=1$, or $\kappa\ge2$ and $\Delta\tau\,\mu^2<\frac14$. Then, there exists a constant $C>0$, independent of $\Delta\tau$, such that*

$$\left(\sum_{m=1}^{M}\Delta\tau\,\|W^m-w(\tau_m,\cdot)\|_{0,D}^2\right)^{\frac12}\le\,C\,(\Delta\tau)^{\theta}\,\|w_0\|_{\dot{\mathbf{H}}^{4\theta-2}},\quad\forall\,w_0\in\dot{\mathbf{H}}^2(D),\quad\forall\,\theta\in[0,1].\tag{4.5}$$

Proof. The estimate (4.5) will be established by interpolation, after proving it for $\theta=1$ and $\theta=0$.

Let $w_0\in\dot{\mathbf{H}}^2(D)$. According to the discussion in the begining of this section, when $\kappa=1$ or $\kappa\geq 2$ and $\Delta\tau\,\mu^2<\frac{1}{4}$, the existence and uniqueness of the time-discrete approximations $(W^m)_{m=0}^M$ is secured. We omit the case $\kappa=1$ since then the operator Λ_{B} is invertible and the proof of (4.5) follows moving along the lines of the proof of Proposition 4.1 in [14], or alternatively moving along the lines of the proof below using the operator T_{B} instead of $\widetilde{T}_{B}$. Here, we will proceed with the proof of (4.5) under the assumption $\Delta\tau\,\mu^2<\frac{1}{4}$, without using somewhere a possible invertibilty of Λ_{B}. In the sequel, we will use the symbol C to denote a generic constant that is independent of Δt and may changes value from one line to the other.

Let $E^m(\cdot):=w(\tau_m,\cdot)-V^m(\cdot)$ for $m=0,\dots,M$ and $\sigma_m:=\int_{\Delta_m}\left[w(\tau_m,\cdot)-w(\tau,\cdot)\right]d\tau$ for $m=1,\dots,M$. Then, combining (1.5) and (4.4), we conclude that

$$\widetilde{T}_{B}(E^m-E^{m-1})+\Delta\tau\,E^m=\Delta\tau\,\mu^2\,\widetilde{T}_{B}E^m+\Big(\sigma_m-\mu^2\,\widetilde{T}_{B}\sigma_m\Big),\quad m=1,\dots,M. \tag{4.6}$$

Now, take the $L^2(D)-$inner product with E^m of both sides of (4.6), to obtain

$$\begin{aligned}\widetilde{\gamma}_{B}(E^m-E^{m-1},E^m)_{0,D}+\Delta\tau\,\|E^m\|_{0,D}^2=&\,\Delta\tau\,\mu^2\,\widetilde{\gamma}_{B}(E^m,E^m)\\&+(\sigma_m-\mu^2\,\widetilde{T}_{B}\sigma_m,E^m)_{0,D},\quad m=1,\dots,M.\end{aligned}\tag{4.7}$$

Using (2.11), (4.7) and (2.15), we arrive at

$$\begin{aligned}\widetilde{\gamma}_{B}(E^m,E^m)-\widetilde{\gamma}_{B}(E^{m-1},E^{m-1})+\Delta\tau\,\|E^m\|_{0,D}^2\leq&\,2\,\Delta\tau\,\mu^2\,\widetilde{\gamma}_{B}(E^m,E^m)\\&+C\,\Delta\tau^{-1}\,\|\sigma_m\|_{0,D}^2,\quad m=1,\dots,M.\end{aligned}\tag{4.8}$$

Since $2\,\Delta\tau\,\mu^2<1$, (4.8) yields

$$\widetilde{\gamma}_{B}(E^m,E^m)\leq\tfrac{1}{1-2\,\mu^2\,\Delta\tau}\,\left[\widetilde{\gamma}_{B}(E^{m-1},E^{m-1})+C\,\Delta\tau^{-1}\,\|\sigma_m\|_{0,D}^2\right],\quad m=1,\dots,M.$$

Then, we apply a simple induction argument and use that $E^0=0$ and $4\,\Delta\tau\,\mu^2<1$, to obtain

$$\begin{aligned}\widetilde{\gamma}_{B}(E^m,E^m)&\leq C\,\Delta\tau^{-1}\sum_{\ell=1}^m\|\sigma_\ell\|_{0,D}^2\,\tfrac{1}{(1-2\,\Delta\tau\,\mu^2)^{m+1-\ell}}\\&\leq C\,e^{4T\mu^2}\,\Delta\tau^{-1}\sum_{\ell=1}^m\|\sigma_\ell\|_{0,D}^2,\quad m=1,\dots,M.\end{aligned}\tag{4.9}$$

Next, we use the Cauchy-Schwarz inequality to bound σ_m as follows:

$$\begin{aligned}\|\sigma_m\|_{0,D}^2&\leq\,C\int_D\left(\int_{\Delta_m}\int_{\Delta_m}|\partial_\tau w(s,x)|\,dsd\tau\right)^2\,dx\\&\leq\,C\,(\Delta\tau)^3\int_{\Delta_m}\|\partial_\tau w(s,\cdot)\|_{0,D}^2\,ds,\quad m=1,\dots,M.\end{aligned}\tag{4.10}$$

Thus, (4.10) and (4.9) yield

$$\widetilde{\gamma}_{B}(E^m,E^m)\leq\,C\,(\Delta\tau)^2\int_0^{\tau_m}\|\partial_\tau w(s,\cdot)\|_{0,D}^2\,ds,\quad m=1,\dots,M. \tag{4.11}$$

Combining (4.8), (4.11) and (4.10), we have

$$\begin{aligned}\widetilde{\gamma}_{B}(E^m,E^m)-\widetilde{\gamma}_{B}(E^{m-1},E^{m-1})+\Delta\tau\,\|E^m\|_{0,D}^2\leq&\,C\,(\Delta\tau)^2\int_{\Delta_m}\|\partial_\tau w(s,\cdot)\|_{0,D}^2\,ds\\&+C\,(\Delta\tau)^3\int_0^{\tau_m}\|\partial_\tau w(s,\cdot)\|_{0,D}^2\,ds\end{aligned}\tag{4.12}$$

for $m=1,\dots,M$. Summing with respect to m from 1 up to M and using the fact that $E^0=0$, (4.12) yields

$$\widetilde{\gamma}_{B}(E^M,E^M)+\sum_{m=1}^M\Delta\tau\,\|E^m\|_{0,D}^2\leq\,C\,(\Delta\tau)^2\int_0^T\|\partial_\tau w(s,\cdot)\|_{0,D}^2\,ds. \tag{4.13}$$

Finally, use (4.13) and (2.16) (with $\beta=0$, $\ell=1$, $p=0$) to obtain

$$\left(\sum_{m=1}^{M}\Delta\tau\,\|E^m\|_{0,D}^2\right)^{\frac{1}{2}}\leq\,C\,\Delta\tau\,\|w_0\|_{\dot{\mathbf{H}}^2},\tag{4.14}$$

which establishes (4.5) for $\theta=1$.

First, we observe that (4.4) is written equivalently as

$$\widetilde{T}_B(W^m-W^{m-1})+\Delta\tau\,W^m=\Delta\tau\,\mu^2\,\widetilde{T}_BW^m,\quad m=1,\dots,M,$$

from which, after taking the $L^2(D)-$inner product with W^m, we obtain

$$\widetilde{\gamma}_B(W^m-W^{m-1},W^m)_{0,D}+\Delta\tau\,\|W^m\|_{0,D}^2=\Delta\tau\,\mu^2\,\widetilde{\gamma}_B(W^m\,W^m),\quad m=1,\dots,M.\tag{4.15}$$

Then, we combine (2.11) and (4.15) to have

$$(1-2\,\Delta\tau\,\mu^2)\,\widetilde{\gamma}_B(W^m,W^m)+2\,\Delta\tau\,\|W^m\|_{0,D}^2\leq\,\widetilde{\gamma}_B(W^{m-1},W^{m-1}),\quad m=1,\dots,M.\tag{4.16}$$

Since $4\,\mu^2\,\Delta\tau<1$, (4.16) yields that

$$\begin{aligned}\widetilde{\gamma}_B(W^m,W^m)&\leq\tfrac{1}{1-2\,\mu^2\,\Delta\tau}\,\widetilde{\gamma}_B(W^{m-1},W^{m-1})\\&\leq e^{4\mu^2\Delta\tau}\,\widetilde{\gamma}_B(W^{m-1},W^{m-1}),\quad m=1,\dots,M,\end{aligned}$$

from which, applying a simple induction argument, we conclude that

$$\max_{0\leq m\leq M}\widetilde{\gamma}_B(W^m,W^m)\leq\,C\,\widetilde{\gamma}_B(w_0,w_0).\tag{4.17}$$

Now, summing with respect to m from 1 up to M, and using (4.17), (4.16) yields

$$\begin{aligned}\sum_{m=1}^{M}\Delta\tau\,\|W^m\|_{0,D}^2&\leq C\,(\widetilde{T}_Bw_0,w_0)_{0,D}\\&\leq\|w_0\|_{-2,D}\,\|\widetilde{T}_Bw_0\|_{2,D}.\end{aligned}\tag{4.18}$$

Thus, using (4.18), (2.15) and (2.4), we obtain

$$\begin{aligned}\left(\sum_{m=1}^{M}\Delta\tau\,\|W^m\|_{0,D}^2\right)^{\frac{1}{2}}&\leq\,C\,\|w_0\|_{-2,D}\\&\leq\,C\,\|w_0\|_{\dot{\mathbf{H}}^{-2}}.\end{aligned}\tag{4.19}$$

In addition we have

$$\begin{aligned}\sum_{m=1}^{M}\Delta\tau\,\|w(\tau_m,\cdot)\|_{0,D}^2&\leq\,\sum_{m=1}^{M}\int_D\left(\int_{\Delta_m}\partial_\tau\big[(\tau-\tau_{m-1})\,w^2(\tau,x)\big]\,d\tau\right)\,dx\\&\leq\,\sum_{m=1}^{M}\int_D\left(\int_{\Delta_m}\big[w^2(\tau,x)+2\,(\tau-\tau_{m-1})\,w_\tau(\tau,x)\,w(\tau,x)\big]\,d\tau\right)\,dx\\&\leq\,\sum_{m=1}^{M}\int_{\Delta_m}\Big[2\,\|w(\tau,\cdot)\|_{0,D}^2+(\tau-\tau_{m-1})^2\,\|w_\tau(\tau,\cdot)\|_{0,D}^2\Big]\,d\tau\\&\leq\,2\int_0^T\Big[\|w(\tau,\cdot)\|_{0,D}^2+\tau^2\,\|w_\tau(\tau,\cdot)\|_{0,D}^2\Big]\,d\tau,\end{aligned}$$

which, along with (2.16) (taking $(\beta,\ell,p)=(0,0,0)$ and $(\beta,\ell,p)=(2,1,0)$) and (2.4), yields

$$\left(\sum_{m=1}^{M}\Delta\tau\,\|w(\tau_m,\cdot)\|_{0,D}^2\right)^{\frac{1}{2}}\leq\,C\,\|w_0\|_{\dot{\mathbf{H}}^{-2}}.\tag{4.20}$$

Thus, the estimate (4.5) for $\theta=0$ follows easily combining (4.19) and (4.20). □

4.2. **The Stochastic Case.** Next theorem combines the convergence result of Proposition 4.1 with a discrete Duhamel's principle in order to prove a discrete in time $L^\infty_t(L^2_P(L^2_x))$ convergence estimate for the time discrete approximations of $\widehat{u}$ (cf. [14], [22]).

Theorem 4.2. *Let $\widehat{u}$ be the solution of (1.6) and $(\widehat{U}^m)_{m=0}^M$ be the time-discrete approximations defined by (4.1)–(4.2). Also, we assume that $\kappa = 1$, or $\kappa \geq 2$ and $\Delta\tau\,\mu^2 < \frac{1}{4}$. Then, there exists a constant $C > 0$, independent of Δt, Δx and $\Delta\tau$, such that*

$$\max_{1\leq m\leq M}\left(\mathbb{E}\left[\|\widehat{U}^m - \widehat{u}(\tau_m,\cdot)\|^2_{0,D}\right]\right)^{\frac{1}{2}} \leq C\,\omega_1(\Delta\tau,\epsilon)\;\Delta\tau^{\frac{1}{8}-\epsilon},\quad \forall\,\epsilon\in(0,\tfrac{1}{8}], \tag{4.21}$$

where $\omega_1(\Delta\tau,\epsilon) := \epsilon^{-\frac{1}{2}} + (\Delta\tau)^\epsilon(1+(\Delta\tau)^{\frac{7}{4}}+(\Delta\tau)^{\frac{3}{4}})^{\frac{1}{2}}$.

Proof. Let $I : L^2(D) \to L^2(D)$ be the identity operator, $\Lambda : L^2(D) \to \dot{\mathbf{H}}^4(D)$ be the inverse elliptic operator $\Lambda := (I + \Delta\tau\,\Lambda_B)^{-1}$ which has Green function $G_\Lambda(x,y) = \sum_{k=1}^\infty \frac{\varepsilon_k(x)\,\varepsilon_k(y)}{1+\Delta\tau\,\lambda_k^2(\lambda_k^2-\mu)}$, i.e. $\Lambda f(x) = \int_D G_\Lambda(x,y)f(y)\,dy$ for $x\in\overline{D}$ and $f\in L^2(D)$. Also, we set $G_\Phi(x,y) := -\partial_y G_\Lambda(x,y) = -\sum_{k=1}^\infty \frac{\varepsilon_k(x)\,\varepsilon_k'(y)}{1+\Delta\tau(\lambda_k^4-\mu\,\lambda_k^2)}$, and define $\Phi : L^2(D)\to\dot{\mathbf{H}}^4(D)$ by $\Phi f(x) := \int_D G_\Phi(x,y)\,f(y)\,dy$ for $f\in L^2(D)$. Also, for $m\in\mathbb{N}$, we denote by $G_{\Lambda\Phi,m}$ the Green function of the operator $\Lambda^{m-1}\Phi$. In the sequel, we will use the symbol C to denote a generic constant that is independent of Δt, $\Delta\tau$ and Δx, and may changes value from one line to the other.

Using (4.2) and a simple induction argument, we conclude that

$$\widehat{U}^m = \sum_{j=1}^m \int_{\Delta_j} \Lambda^{m-j}\Phi\widehat{W}(\tau,\cdot)\,d\tau,\quad m=1,\ldots,M,$$

which is written, equivalently, as follows:

$$\widehat{U}^m(x) = \int_0^{\tau_m}\!\!\int_D \widehat{\mathcal{K}}_m(\tau;x,y)\,\widehat{W}(\tau,y)\,dyd\tau,\quad \forall\,x\in\overline{D},\quad m=1,\ldots,M, \tag{4.22}$$

where $\widehat{\mathcal{K}}_m(\tau;x,y) := \sum_{j=1}^m \mathcal{X}_{\Delta_j}(\tau)\,G_{\Lambda\Phi,m-j+1}(x,y)$, $\quad\forall\,\tau\in[0,T]$, $\;\forall\,x,y\in D$.

Let $m\in\{1,\ldots,M\}$ and $\mathcal{E}^m := \mathbb{E}\big[\|\widehat{U}^m - \widehat{u}(\tau_m,\cdot)\|^2_{0,D}\big]$. First, we use (4.22), (1.7), (2.9), (2.6), (2.5) and (2.8), to obtain

$$\begin{aligned}
\mathcal{E}^m &= \mathbb{E}\left[\int_D\left(\int_0^T\!\!\int_D \mathcal{X}_{(0,\tau_m)}(\tau)\left[\widehat{\mathcal{K}}_m(\tau;x,y)-\Psi(\tau_m-\tau;x,y)\right]\widehat{W}(\tau,y)\,dyd\tau\right)^2 dx\right]\\
&\leq \int_0^{\tau_m}\left(\int_D\!\int_D\left[\widehat{\mathcal{K}}_m(\tau;x,y)-\Psi(\tau_m-\tau;x,y)\right]^2 dydx\right)d\tau\\
&\leq \sum_{\ell=1}^m\int_{\Delta_\ell}\left(\int_D\!\int_D\left[G_{\Lambda\Phi,m-\ell+1}(x,y)-\Psi(\tau_m-\tau;x,y)\right]^2 dydx\right)d\tau.
\end{aligned}$$

Now, we introduce the splitting

$$\sqrt{\mathcal{E}^m} \leq \sqrt{\mathcal{B}_1^m} + \sqrt{\mathcal{B}_2^m}, \tag{4.23}$$

where

$$\begin{aligned}
\mathcal{B}_1^m &:= \sum_{\ell=1}^m\int_{\Delta_\ell}\left(\int_D\!\int_D\left[G_{\Lambda\Phi,m-\ell+1}(x,y)-\Psi(\tau_m-\tau_{\ell-1};x,y)\right]^2 dydx\right)d\tau,\\
\mathcal{B}_2^m &:= \sum_{\ell=1}^m\int_{\Delta_\ell}\left(\int_D\!\int_D\left[\Psi(\tau_m-\tau_{\ell-1};x,y)-\Psi(\tau_m-\tau;x,y)\right]^2 dydx\right)d\tau.
\end{aligned}$$

By the definition of the Hilbert-Schmidt norm, we have

$$
\begin{aligned}
\mathcal{B}_1^m &\leq \Delta\tau \sum_{\ell=1}^{m}\sum_{k=1}^{\infty}\int_D \left(\int_D \left[G_{\Lambda\Phi,m-\ell+1}(x,y)\varphi_k(y)\,dy - \int_D \Psi(\tau_m-\tau_{\ell-1};x,y)\varphi_k(y)\,dy\right)^2\,dx\right.\\
&\leq \sum_{k=1}^{\infty}\left(\sum_{\ell=1}^{m}\Delta\tau\,\|\Lambda^{m-\ell}\Phi\varphi_k - \mathcal{S}(\tau_m-\tau_{\ell-1})\varphi_k'\|_{0,D}^2\right)\\
&\leq \sum_{k=1}^{\infty}\left(\sum_{\ell=1}^{m}\Delta\tau\,\|\Lambda^{m-\ell+1}\varphi_k' - \mathcal{S}(\tau_m-\tau_{\ell-1})\varphi_k'\|_{0,D}^2\right)\\
&\leq \sum_{k=1}^{\infty}\lambda_k^2\left(\sum_{\ell=1}^{m}\Delta\tau\,\|\Lambda^{\ell}\varepsilon_k - \mathcal{S}(\tau_\ell)\varepsilon_k\|_{0,D}^2\right).
\end{aligned}
$$

Let $\theta\in[0,\frac{1}{8})$. Using the deterministic error estimate (4.5) and (2.10), we obtain

$$
\begin{aligned}
\sqrt{\mathcal{B}_1^m} &\leq C\,(\Delta\tau)^{\theta}\left(\sum_{k=1}^{\infty}\lambda_k^2\,\|\varepsilon_k\|_{\dot{\mathbf{H}}^{4\theta-2}}^2\right)^{\frac{1}{2}}\\
&\leq C\,(\Delta\tau)^{\theta}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^{1+8\,(\frac{1}{8}-\theta)}}\right)^{\frac{1}{2}}\\
&\leq C\,\frac{1}{\frac{1}{8}-\theta}\,(\Delta\tau)^{\theta}.
\end{aligned}
\tag{4.24}
$$

Using, again, the definition of the Hilbert-Schmidt norm we have

$$
\begin{aligned}
\mathcal{B}_2^m &= \sum_{k=1}^{\infty}\sum_{\ell=1}^{m}\int_{\Delta_\ell}\|\mathcal{S}(\tau_m-\tau_{\ell-1})\varphi_k' - \mathcal{S}(\tau_m-\tau)\varphi_k'\|_{0,D}^2\,d\tau\\
&= \sum_{k=1}^{\infty}\lambda_k^2\sum_{\ell=1}^{m}\int_{\Delta_\ell}\|\mathcal{S}(\tau_m-\tau_{\ell-1})\varepsilon_k - \mathcal{S}(\tau_m-\tau)\varepsilon_k\|_{0,D}^2\,d\tau
\end{aligned}
\tag{4.25}
$$

Observing that $\mathcal{S}(t)\varepsilon_k = e^{-\lambda_k^2(\lambda_k^2-\mu)t}\,\varepsilon_k$ for $t\geq 0$, (4.25) yields

$$
\begin{aligned}
\mathcal{B}_2^m &= \sum_{k=1}^{\infty}\lambda_k^2\sum_{\ell=1}^{m}\int_{\Delta_\ell}\left(\int_D\left[e^{-(\lambda_k^4-\mu\,\lambda_k^2)(\tau_m-\tau_{\ell-1})} - e^{-(\lambda_k^4-\mu\,\lambda_k^2)(\tau_m-\tau)}\right]^2\varepsilon_k^2(x)\,dx\right)\,d\tau\\
&= \sum_{k=1}^{\infty}\lambda_k^2\sum_{\ell=1}^{m}\int_{\Delta_\ell}e^{-2(\lambda_k^4-\mu\,\lambda_k^2)(\tau_m-\tau)}\left[1-e^{-(\lambda_k^4-\mu\,\lambda_k^2)(\tau-\tau_{\ell-1})}\right]^2\,d\tau\\
&\leq \mathcal{B}_{2,1}^m + \mathcal{B}_{2,2}^m,
\end{aligned}
\tag{4.26}
$$

where

$$
\begin{aligned}
\mathcal{B}_{2,1}^m &:= \sum_{k=1}^{\kappa}\lambda_k^2\sum_{\ell=1}^{m}\int_{\Delta_\ell}e^{-2\lambda_k^2(\lambda_k^2-\mu)(\tau_m-\tau)}\left[1-e^{-(\lambda_k^4-\mu\,\lambda_k^2)(\tau-\tau_{\ell-1})}\right]^2\,d\tau,\\
\mathcal{B}_{2,2}^m &:= \sum_{k=\kappa+1}^{\infty}\lambda_k^2\sum_{\ell=1}^{m}\int_{\Delta_\ell}e^{-2\lambda_k^2(\lambda_k^2-\mu)(\tau_m-\tau)}\left[1-e^{-(\lambda_k^4-\mu\,\lambda_k^2)(\tau-\tau_{\ell-1})}\right]^2\,d\tau.
\end{aligned}
$$

First, we estimate $\mathcal{B}_{2,1}^m$ and $\mathcal{B}_{2,2}^m$ as follows

$$
\begin{aligned}
\mathcal{B}_{2,2}^m &\leq \sum_{k=\kappa+1}^{\infty} \lambda_k^2 \left(1-e^{-\lambda_k^2(\lambda_k^2-\mu)\,\Delta\tau}\right)^2 \left[\int_0^{\tau_m} e^{-2(\lambda_k^4-\mu\,\lambda_k^2)(\tau_m-\tau)}\,d\tau\right]\\
&\leq \tfrac{1}{2} \sum_{k=\kappa+1}^{\infty} \tfrac{1-e^{-2\lambda_k^2(\lambda_k^2-\mu)\,\Delta\tau}}{\lambda_k^2-\mu}\\
&\leq \tfrac{(\kappa+1)^2}{2(1+2\kappa)} \sum_{k=\kappa+1}^{\infty} \tfrac{1-e^{-2\lambda_k^2(\lambda_k^2-\mu)\,\Delta\tau}}{\lambda_k^2}\\
&\leq C \sum_{k=1}^{\infty} \tfrac{1-e^{-c_0\,\lambda_k^4\,\Delta\tau}}{\lambda_k^2}
\end{aligned}
\tag{4.27}
$$

with $c_0 = \frac{2(1+2\kappa)}{(\kappa+1)^2}$, and

$$
\begin{aligned}
\mathcal{B}_{2,1}^m &\leq C \sum_{k=1}^{\kappa}\sum_{\ell=1}^{m} \int_{\Delta_\ell} \left[1-e^{-(\lambda_k^4-\mu\,\lambda_k^2)(\tau-\tau_{\ell-1})}\right]^2 d\tau\\
&\leq C \sum_{k=1}^{\kappa}\sum_{\ell=1}^{m} \int_{\Delta_\ell} \left[(\lambda_k^4-\mu\,\lambda_k^2)(\tau-\tau_{\ell-1})\right]^2 d\tau\\
&\leq C\,(\Delta\tau)^2.
\end{aligned}
\tag{4.28}
$$

Finally, we combine (4.26), (4.27), (4.28) and (3.17), to obtain

$$
\sqrt{\mathcal{B}_2^m} \leq C \left(1+(\Delta\tau)^{\frac{3}{4}}+(\Delta\tau)^{\frac{7}{4}}\right)^{\frac{1}{2}} (\Delta\tau)^{\frac{1}{8}}.
\tag{4.29}
$$

The estimate (4.21) follows by (4.23), (4.24) and (4.29). □ □

5. Convergence of the Fully-Discrete Approximations

To get an error estimate for the fully-discrete approximations of $\widehat{u}$ defined by (1.8)–(1.9), we proceed by comparing them with their time-discrete approximations defined by (4.1)–(4.2) and using a discrete Duhamel principle (cf. [14], [22]).

5.1. **The Deterministic Case.** The Backward Euler finite element approximations of the solution to (1.5) are defined as follows: first, set

$$
W_h^0 := P_h w_0,
\tag{5.1}
$$

and then, for $m=1,\dots,M$, find $W_h^m \in M_h^r$ such that

$$
W_h^m - W_h^{m-1} + \Delta\tau\,\Lambda_{B,h} W_h^m = 0,
\tag{5.2}
$$

which is possible when $\mu^2\,\Delta\tau < 4$.

Next, we derive a discrete in time $L_t^2(L_x^2)$ estimate for the error approximating the Backward Euler time-discrete approximations of the solution to (1.5) defined in (4.3)-(4.4), by the Backward Euler finite element approximations defined in (5.1)-(5.2). The main difference with the case $\mu=0$ which has been considered in [14], is that, our assumption (1.2) on μ, can not ensure the coerciveness of the discrete elliptic operator $\Lambda_{B,h}$.

Theorem 5.1. *Let $r=2$ or 3, w be the solution to the problem (1.5), $(W^m)_{m=0}^M$ be the time-discrete approximations of w defined in (4.3)-(4.4), and $(W_h^m)_{m=0}^M \subset M_h^r$ be the fully-discrete approximations of w defined in (5.1)-(5.2). Also, we assume that $\mu^2\,\Delta\tau < \frac{1}{4}$. If $w_0 \in \dot{\mathbf{H}}^2(D)$, then, there exists a nonnegative constant $\widehat{c}_1$, independent of h and $\Delta\tau$, such that*

$$
\left(\sum_{m=1}^{M} \Delta\tau\,\|W^m - W_h^m\|_{0,D}^2\right)^{\frac{1}{2}} \leq \widehat{c}_1\; h^{\ell_\star(r)\,\theta}\; \|w_0\|_{\dot{\mathbf{H}}^{\xi_\star(r,\theta)}}, \quad \forall\,\theta\in[0,1],
\tag{5.3}
$$

where

$$\ell_\star(r) := \begin{cases} 2 & \text{if} \quad r=2 \\ 4 & \text{if} \quad r=3 \end{cases} \quad \textit{and} \quad \xi_\star(r,\theta) := (r+1)\,\theta - 2. \tag{5.4}$$

Proof. The error estimate (5.3) follows by interpolation, after showing that holds for $\theta = 0$ and $\theta = 1$. In the sequel, we will use the symbol C to denote a generic constant that is independent of $\Delta\tau$ and h, and may changes value from one line to the other.

Let $E^m := W_h^m - W^m$ for $m = 0, \dots, M$. First, use (5.2) and (4.4) to obtain

$$W_h^m - W_h^{m-1} + \Delta\tau\,\widetilde{\Lambda}_{B,h} W_h^m = \Delta\tau\,\mu^2\,W_h^m, \tag{5.5}$$

$$W^m - W^{m-1} + \Delta\tau\,\widetilde{\Lambda}_B W^m = \Delta\tau\,\mu^2\,W^m \tag{5.6}$$

for $m = 1, \dots, M$. Then, combine (5.5) and (5.6), to get the following error equation

$$\widetilde{T}_{B,h}(E^m - E^{m-1}) + \Delta\tau\,E^m = \Delta\tau\,\mu^2\,\widetilde{T}_{B,h}E^m - \Delta\tau\,(\widetilde{T}_B - \widetilde{T}_{B,h})\widetilde{\Lambda}_B W^m, \quad m = 1, \dots, M. \tag{5.7}$$

Taking the $L^2(D)-$inner product with E^m of both sides of (5.7), it follows that

$$\begin{aligned} \widetilde{\gamma}_{B,h}(E^m - E^{m-1}, E^m) + \Delta\tau\,\|E^m\|_{0,D}^2 &= \Delta\tau\,\mu^2\,\widetilde{\gamma}_{B,h}(E^m, E^m) \\ &\quad - \Delta\tau\,((\widetilde{T}_B - \widetilde{T}_{B,h})\widetilde{\Lambda}_B W^m, E^m)_{0,D}, \quad m = 1, \dots, M, \end{aligned}$$

from which, after using (2.11), we conclude that

$$\begin{aligned} \widetilde{\gamma}_{B,h}(E^m, E^m) + \Delta\tau\,\|E^m\|_{0,D}^2 &\leq \widetilde{\gamma}_{B,h}(E^{m-1}, E^{m-1}) + 2\,\Delta\tau\,\mu^2\,\widetilde{\gamma}_{B,h}(E^m, E^m) \\ &\quad + \Delta\tau\,\|(\widetilde{T}_B - \widetilde{T}_{B,h})\widetilde{\Lambda}_B W^m\|_{0,D}^2, \quad m = 1, \dots, M. \end{aligned} \tag{5.8}$$

Since $2\Delta\tau\mu^2 < 1$, (5.8) yields

$$\widetilde{\gamma}_{B,h}(E^m, E^m) \leq \tfrac{1}{1 - 2\,\Delta\tau\,\mu^2}\,\left[\widetilde{\gamma}_{B,h}(E^{m-1}, E^{m-1}) + \Delta\tau\,\|(\widetilde{T}_B - \widetilde{T}_{B,h})\widetilde{\Lambda}_B W^m\|_{0,D}^2\right] \tag{5.9}$$

for $m = 1, \dots, M$. Applying a simple induction argument based on (5.8) and then using that $4\Delta\tau\mu^2 < 1$, we get

$$\max_{0 \leq m \leq M} \widetilde{\gamma}_{B,h}(E^m, E^m) \leq C\,\left[\widetilde{\gamma}_{B,h}(E^0, E^0) + \Delta\tau \sum_{\ell=1}^{M} \|(\widetilde{T}_B - \widetilde{T}_{B,h})\widetilde{\Lambda}_B W^\ell\|_{0,D}^2\right]. \tag{5.10}$$

Summing with respect to m from 1 up to M, using (5.10) and observing that $\widetilde{T}_{B,h}E^0 = 0$, (5.8) gives

$$\sum_{m=1}^{M} \Delta\tau\,\|E^m\|_{0,D}^2 \leq C\,\sum_{m=1}^{M} \Delta\tau\,\|(\widetilde{T}_B - \widetilde{T}_{B,h})\widetilde{\Lambda}_B W^m\|_{0,D}^2. \tag{5.11}$$

Let $r = 3$. Then, by (2.22), (5.11) and the Poincaré-Friedrich inequality, we obtain

$$\begin{aligned} \left(\sum_{m=1}^{M} \Delta\tau\,\|E^m\|_{0,D}^2\right)^{\frac{1}{2}} &\leq C\,h^4\,\left(\sum_{m=1}^{M} \Delta\tau\,\|\widetilde{\Lambda}_B W^m\|_{0,D}^2\right)^{\frac{1}{2}} \\ &\leq C\,h^4\,\left[\sum_{m=1}^{M} \Delta\tau\,\left(\|\partial_x^4 W^m\|_{0,D}^2 + \|\partial_x^2 W^m\|_{0,D}^2 + \|\partial_x^1 W^m\|_{0,D}^2\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{5.12}$$

Taking the $L^2(D)-$inner product of (4.4) with $\partial^4 W^m$ and then integrating by parts, we obtain

$$(\partial^2 W^m - \partial^2 W^{m-1}, \partial^2 W^m)_{0,D} + \Delta\tau\,\|\partial^4 W^m\|_{0,D}^2 + \mu\,\Delta\tau\,(\partial^2 W^m, \partial^4 W^m)_{0,D} = 0, \quad m = 1, \dots, M. \tag{5.13}$$

Using (2.11), (5.13) and the Cauchy-Schwarz inequality we obtain

$$\|\partial^2 W^m\|_{0,D}^2 + 2\,\Delta\tau\,\|\partial^4 W^m\|_{0,D}^2 \leq \|\partial^2 W^{m-1}\|_{0,D}^2 + 2\,\mu\,\Delta\tau\,\|\partial^2 W^{m-1}\|_{0,D}\,\|\partial^4 W^m\|_{0,D}, \quad m = 1, \dots, M,$$

which, after using the geometric mean inequality, yields

$$\|\partial^2 W^m\|_{0,D}^2 + \Delta\tau\,\|\partial^4 W^m\|_{0,D}^2 \leq \|\partial^2 W^{m-1}\|_{0,D}^2 + \Delta\tau\,\mu^2\,\|\partial^2 W^m\|_{0,D}^2, \quad m = 1, \dots, M. \tag{5.14}$$

Since $2\,\mu^2\,\Delta\tau<1$, from (5.14) follows that

$$\begin{aligned}\|\partial^2 W^m\|^2_{0,D} &\leq \tfrac{1}{1-\mu^2\,\Delta\tau}\,\|\partial^2 W^{m-1}\|^2_{0,D}\\ &\leq e^{2\mu^2\,\Delta\tau}\,\|\partial^2 W^{m-1}\|^2_{0,D}, \quad m=1,\dots,M,\end{aligned}$$

from which, applying a simple induction argument, we conclude that

$$\max_{0\leq m\leq M}\|\partial^2 W^m\|^2_{0,D}\leq\,C\,\|w_0\|^2_{2,D}. \tag{5.15}$$

Next, sum both side of (5.14) with respect to m, from 1 up to M, and use (5.15) to conclude that

$$\sum_{m=1}^{M}\Delta\tau\,\|\partial^4 W^m\|^2_{0,D}\leq\,C\,\|w_0\|^2_{2,D}. \tag{5.16}$$

Taking the $L^2(D)-$inner product of (4.4) with $\partial^2 W^m$, and then integrating by parts, it follows that

$$(\partial W^m-\partial W^{m-1},\partial W^m)_{0,D}+\Delta\tau\,\|\partial^3 W^m\|^2_{0,D}+\mu\,\Delta\tau\,(\partial W^m,\partial^3 W^m)_{0,D}=0, \quad m=1,\dots,M. \tag{5.17}$$

Using (2.11), (5.17), the Cauchy-Schwarz inequality and the geometric mean inequality, we obtain

$$\|\partial W^m\|^2_{0,D}+\Delta\tau\,\|\partial^3 W^m\|^2_{0,D}\leq\|\partial W^{m-1}\|^2_{0,D}+\Delta\tau\,\mu^2\,\|\partial W^m\|^2_{0,D}, \quad m=1,\dots,M.$$

Since $2\,\mu^2\,\Delta\tau<1$, proceeding as in obtaining (5.15) and (5.16) from (5.14), we arrive at

$$\max_{0\leq m\leq M}\|\partial W^m\|^2_{0,D}+\sum_{m=1}^{M}\Delta\tau\,\|\partial^3 W^m\|^2_{0,D}\leq\,C\,\|w_0\|^2_{1,D}. \tag{5.18}$$

Thus, combining (5.12), (5.16), (5.15), (5.18) and (2.3), we obtain

$$\left(\sum_{m=1}^{M}\Delta\tau\,\|E^m\|^2_{0,D}\right)^{\frac12}\leq\,C\,h^4\,\|w_0\|_{\dot{\mathbf{H}}^2}. \tag{5.19}$$

Let $r=2$. Then, by (2.22), (5.11) and the Poincaré-Friedrich inequality, we obtain

$$\begin{aligned}\left(\sum_{m=1}^{M}\Delta\tau\,\|E^m\|^2_{0,D}\right)^{\frac12} &\leq C\,h^2\,\left(\sum_{m=1}^{M}\Delta\tau\,\|\widetilde{\Lambda}_B W^m\|^2_{-1,D}\right)^{\frac12}\\ &\leq C\,h^2\,\left[\sum_{m=1}^{M}\Delta\tau\,\left(\,\|\partial^3 W^m\|^2_{0,D}+\|\partial W^m\|^2_{0,D}\,\right)\right]^{\frac12}.\end{aligned} \tag{5.20}$$

Combining, now, (5.20), (5.18) and (2.3), we obtain

$$\left(\sum_{m=1}^{M}\Delta\tau\,\|E^m\|^2_{0,D}\right)^{\frac12}\leq\,C\,h^2\,\|w_0\|_{\dot{\mathbf{H}}^1}. \tag{5.21}$$

Thus, relations (5.19) and (5.21) yield (5.3) and (5.4) for $\theta=1$.

Since $\mu^2\,\Delta\tau<1$, using (5.5), we have

$$\widetilde{T}_{B,h}(W_h^m-W_h^{m-1})+\Delta\tau\,W_h^m=\Delta\tau\,\mu^2\,\widetilde{T}_{B,h}W_h^m, \quad m=1,\dots,M,$$

from which, after taking the $L^2(D)-$inner product with W_h^m, we obtain

$$\widetilde{\gamma}_{B,h}(W_h^m-W_h^{m-1},W_h^m)_{0,D}+\Delta\tau\,\|W_h^m\|^2_{0,D}=\Delta\tau\,\mu^2\,\widetilde{\gamma}_{B,h}(W_h^m,W_h^m), \quad m=1,\dots,M. \tag{5.22}$$

Then we combine (5.22) with (2.11) to have

$$(1-2\,\Delta\tau\,\mu^2)\,\widetilde{\gamma}_{B,h}(W_h^m,W_h^m)+2\,\Delta\tau\,\|W_h^m\|^2_{0,D}\leq\,\widetilde{\gamma}_{B,h}(W_h^{m-1},W_h^{m-1}), \quad m=1,\dots,M. \tag{5.23}$$

Since $4\,\mu^2\,\Delta\tau<1$, (5.23) yields that

$$\begin{aligned}\widetilde{\gamma}_{B,h}(W_h^m,W_h^m) &\leq \tfrac{1}{1-2\,\mu^2\,\Delta\tau}\,\widetilde{\gamma}_{B,h}(W_h^{m-1},W_h^{m-1})\\ &\leq e^{4\mu^2\,\Delta\tau}\,\widetilde{\gamma}_{B,h}(W_h^{m-1},W_h^{m-1}), \quad m=1,\dots,M,\end{aligned}$$

from which, applying a simple induction argument, we conclude that

$$\max_{0\le m\le M} \widetilde{\gamma}_{B,h}(W_h^m, W_h^m) \le C\,\widetilde{\gamma}_{B,h}(W_h^0, W_h^0). \tag{5.24}$$

Summing with respect to m from 1 up to M, and using (5.24), (5.23) gives

$$\Delta\tau \sum_{m=1}^{M} \|W_h^m\|_{0,D}^2 \le C\,\widetilde{\gamma}_{B,h}(W_h^0, W_h^0)_{0,D}. \tag{5.25}$$

Finally, using (5.25), (2.28) and (2.4) we obtain

$$\begin{aligned}\sum_{m=1}^{M} \Delta\tau\,\|W_h^m\|_{0,D}^2 &\le C\,(\widetilde{T}_{B,h} w_0, w_0)_{0,D}\\ &\le C\,\|w_0\|_{-2,D}^2\\ &\le C\,\|w_0\|_{\dot{\mathbf{H}}^{-2}}^2.\end{aligned} \tag{5.26}$$

Finally, combine (5.26) with (4.19) to get

$$\left(\sum_{m=1}^{M} \Delta\tau\,\|W^m - W_h^m\|_{0,D}^2\right)^{\frac{1}{2}} \le C\,\|w_0\|_{\dot{\mathbf{H}}^{-2}},$$

which yields (5.3) and (5.4) for $\theta = 0$. ☐

5.2. **The Stochastic Case.** Our first step is to show the existence of a Green function for the solution operator of a discrete elliptic problem.

Lemma 5.1. *Let $r = 2$ or 3, $\epsilon > 0$ with $\mu^2\epsilon < 4$, $f \in L^2(D)$ and $\psi_h \in M_h^r$ such that*

$$\psi_h + \epsilon\,\Lambda_{B,h}\psi_h = P_h f. \tag{5.27}$$

Then there exists a function $A_{\epsilon,h} \in H^2(D\times D)$ such that $A_{\epsilon,h}\big|_{\partial(D\times D)} = 0$ and

$$\psi_h(x) = \int_D A_{h,\epsilon}(x,y)\,f(y)\,dy \quad \forall\,x \in \overline{D} \tag{5.28}$$

and $A_{h,\epsilon}(x,y) = A_{h,\epsilon}(y,x)$ for $x, y \in \overline{D}$.

Proof. Let $\delta_{\epsilon,h} : M_h^r \times M_h^r \to \mathbb{R}$ be the inner product on M_h^r given by

$$\begin{aligned}\delta_{\epsilon,h}(\phi,\chi) &:= \epsilon\,(\Lambda_{B,h}\phi,\chi)_{0,D} + (\phi,\chi)_{0,D}\\ &= \epsilon\,(\phi'',\chi'')_{0,D} + \epsilon\,\mu\,(\phi'',\chi)_{0,D} + (\phi,\chi)_{0,D}, \quad \forall\,\phi,\chi \in M_h^r.\end{aligned}$$

We can construct a basis $(\chi_j)_{j=1}^{n_h}$ of M_h^r which is $L^2(D)-$orthonormal, i.e., $(\chi_i,\chi_j)_{0,D} = \delta_{ij}$ for $i, j = 1,\dots,n_h$, and $\delta_{\epsilon,h}-$orthogonal, i.e., there exist $(\lambda_{\epsilon,h,\ell})_{\ell=1}^{n_h} \subset (0,+\infty)$ such that $\delta_{\epsilon,h}(\chi_i,\chi_j) = \lambda_{\epsilon,h,i}\,\delta_{ij}$ for $i, j = 1,\dots,n_h$ (see Section 8.7 in [9]). Thus, there are $(\mu_j)_{j=1}^{n_h} \subset \mathbb{R}$ such that $\psi_h = \sum_{j=1}^{n_h} \mu_j\,\chi_j$, and (5.27) is equivalent to $\mu_i = \frac{1}{\lambda_{\epsilon,h,i}}\,(f,\chi_i)_{0,D}$ for $i = 1,\dots,n_h$. Finally, we obtain (5.28) with $A_{h,\epsilon}(x,y) = \sum_{j=1}^{n_h} \frac{\chi_j(x)\chi_j(y)}{\lambda_{\epsilon,h,j}}$. ☐

Our second step is to compare, in a discrete in time $L_t^\infty(L_P^2(L_x^2))$ norm, the Backward Euler time-discrete approximations of $\widehat{u}$ with the Backward Euler finite element approximations of $\widehat{u}$.

Proposition 5.2. *Let $r = 2$ or 3, $\widehat{u}$ be the solution of the problem (1.6), $(\widehat{U}_h^m)_{m=0}^M$ be the Backward Euler finite element approximations of $\widehat{u}$ defined in (1.8)-(1.9), and $(\widehat{U}^m)_{m=0}^M$ be the Backward Euler time-discrete approximations of $\widehat{u}$ defined in (4.1)-(4.2). Also, we assume that $\mu^2\,\Delta\tau \le \frac{1}{4}$. Then, there exists a nonnegative constant $\widehat{c}_2$, independent of Δx, Δt, h and $\Delta\tau$, such that*

$$\max_{0\le m\le M}\left(\mathbb{E}\left[\left\|\widehat{U}_h^m - \widehat{U}^m\right\|_{0,D}^2\right]\right)^{\frac{1}{2}} \le \widehat{c}_2\,\epsilon^{-\frac{1}{2}}\;h^{\nu(r)-\epsilon}, \quad \forall\,\epsilon \in (0,\nu(r)], \tag{5.29}$$

where

$$\nu(r) := \begin{cases} \frac{1}{3} & \text{if} \quad r=2 \\ \frac{1}{2} & \text{if} \quad r=3 \end{cases}. \tag{5.30}$$

Proof. Let $I : L^2(D) \to L^2(D)$ be the identity operator and $\Lambda_h : L^2(D) \to M_h^r$ be the inverse discrete elliptic operator given by $\Lambda_h := (I + \Delta\tau\, \Lambda_{B,h})^{-1} P_h$, having a Green function $G_{\Lambda_h} = A_{h,\Delta\tau}$ according to Lemma 5.1 and taking into account that $\mu^2\,\Delta\tau < 4$. Also, we define an operator $\Phi_h : L^2(D) \to M_h^r$ by $(\Phi_h f)(x) := \int_D G_{\Phi_h}(x,y)\, f(y)\, dy$ for $f \in L^2(D)$ and $x \in \overline{D}$, where $G_{\Phi_h}(x,y) = -\partial_y G_{\Lambda_h}(x,y)$. Then, we have that $\Lambda_h f' = \Phi_h f$ for all $f \in H^1(D)$. Also, for $\ell \in \mathbb{N}$, we denote by $G_{\Lambda_h,\Phi_h,\ell}$ the Green function of $\Lambda_h^\ell \Phi_h$. In the sequel, we will use the symbol C to denote a generic constant that is independent of Δt, Δx, h and $\Delta\tau$, and may changes value from one line to the other.

Applying, an induction argument, from (1.9) we conclude that

$$\widehat{U}_h^m = \sum_{j=1}^{m} \int_{\Delta_j} \Lambda_h^{m-j} \Phi_h \widehat{W}(\tau,\cdot)\, d\tau, \quad m = 1, \dots, M,$$

which is written, equivalently, as follows:

$$\widehat{U}_h^m(x) = \int_0^{\tau_m}\!\!\int_D \widehat{\mathcal{D}}_{h,m}(\tau; x, y)\, \widehat{W}(\tau, y)\, dy d\tau \quad \forall\, x \in \overline{D}, \quad m = 1, \dots, M, \tag{5.31}$$

where $\widehat{\mathcal{D}}_{h,m}(\tau; x, y) := \sum_{j=1}^m \mathcal{X}_{\Delta_j}(\tau)\, G_{\Lambda_h,\Phi_h,m-j}(x,y) \quad \forall\, \tau \in [0,T], \ \ \forall\, x, y \in D$. Using (4.22), (5.31), the Itô-isometry property of the stochastic integral, (2.5) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \mathbb{E}\left[\|\widehat{U}^m - \widehat{U}_h^m\|_{0,D}^2\right] &\le \int_0^{\tau_m} \Big(\int_D\!\int_D \big[\widehat{\mathcal{K}}_m(\tau; x, y) - \widehat{\mathcal{D}}_{h,m}(\tau; x, y)\big]^2\, dy dx \Big)\, d\tau \\ &\le \sum_{j=1}^m \int_{\Delta_j} \|\Lambda^{m-j}\Phi - \Lambda_h^{m-j}\Phi_h\|_{\mathrm{HS}}^2\, d\tau, \quad m = 1, \dots, M, \end{aligned}$$

where Λ and Φ are the operators defined in the proof of Theorem 4.2. Now, we use the definition of the Hilbert-Schmidt norm and the deterministic error estimate (5.3), to obtain

$$\begin{aligned} \mathbb{E}\left[\|\widehat{U}^m - \widehat{U}_h^m\|_{0,D}^2\right] &\le \sum_{j=1}^m \Delta\tau \left[\sum_{k=1}^\infty \|\Lambda^{m-j}\Phi\varphi_k - \Lambda_h^{m-j}\Phi_h\varphi_k\|_{0,D}^2 \right] \\ &\le \sum_{k=1}^\infty \left[\sum_{\ell=1}^m \Delta\tau\, \|\Lambda^\ell \varphi_k' - \Lambda_h^\ell \varphi_k'\|_{0,D}^2 \right] \\ &\le \sum_{k=1}^\infty \lambda_k^2 \left[\sum_{\ell=1}^m \Delta\tau\, \|\Lambda^\ell \varepsilon_k - \Lambda_h^\ell \varepsilon_k\|_{0,D}^2 \right] \\ &\le C\, h^{2\,\ell_\star(r)\,\theta} \sum_{k=1}^\infty \lambda_k^2\, \|\varepsilon_k\|_{\dot{\mathbf{H}}^{\xi_\star(r,\theta)}}^2, \quad m = 1, \dots, M, \quad \forall\, \theta \in [0,1]. \end{aligned}$$

Thus, we arrive at

$$\max_{1 \le m \le M} \Big(\mathbb{E}\left[\|\widehat{U}^m - \widehat{U}_h^m\|_{0,D}^2\right] \Big)^{\frac{1}{2}} \le C\, h^{\ell_\star(r)\,\theta} \left(\sum_{k=1}^\infty \lambda_k^{-\left[1 + \frac{2\,(r+1)}{\ell_\star(r)}\,(\nu(r) - \ell_\star(r)\,\theta)\right]} \right)^{\frac{1}{2}}, \quad \forall\, \theta \in [0,1]. \tag{5.32}$$

It is easily seen that the series in the right hand side of (5.32) convergences iff $\nu(r) > \ell_\star(r)\,\theta$. Thus, setting $\epsilon = \nu(r) - \ell_\star(r)\,\theta$, requiring $\epsilon \in (0, \nu(r)]$, and combining (5.32) and (2.10), we arrive at the estimate (5.29). $\square$

The available error estimates allow us to conclude a discrete in time $L_t^\infty(L_P^2(L_x^2))$ convergence of the Backward Euler fully-discrete approximations of $\widehat{u}$.

Theorem 5.3. *Let $r=2$ or 3, $\nu(r)$ be defined by* (5.30), *$\widehat{u}$ be the solution of problem* (1.6), *and $(\widehat{U}_h^m)_{m=0}^M$ be the Backward Euler finite element approximations of $\widehat{u}$ constructed by* (1.8)-(1.9). *Then, there exists a nonnegative constant C, independent of h, $\Delta\tau$, Δt and Δx, such that: if $\mu^2\,\Delta\tau\le\frac{1}{4}$, then*

$$\max_{0\le m\le M}\left\{\mathbb{E}\left[\|\widehat{U}_h^m-\widehat{u}(\tau_m,\cdot)\|_{0,D}^2\right]\right\}^{\frac{1}{2}}\le C\left[\omega_*(\Delta\tau,\epsilon_1)\;\Delta\tau^{\frac{1}{8}-\epsilon_1}+\epsilon_2^{-\frac{1}{2}}\;\;h^{\nu(r)-\epsilon_2}\right]$$

forall $\epsilon_1\in(0,\frac{1}{8}]$ and $\epsilon_2\in(0,\nu(r)]$, where $\omega_*(\Delta\tau,\epsilon_1):=\epsilon_1^{-\frac{1}{2}}+(\Delta\tau)^{\epsilon_1}(1+(\Delta\tau)^{\frac{7}{4}}+(\Delta\tau)^{\frac{3}{4}})^{\frac{1}{2}}$.

Proof. The estimate is a simple consequence of the error bounds (5.29) and (4.21). □

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Appendix A.

Let $t>0$ and $\mu_k:=\lambda_k^2(\lambda_k^2-\mu)$ for $k\in\mathbb{N}$. First, we recal that $\mathcal{S}(t)w_0=\sum_{k=1}^{\infty}e^{-\mu_k\,t}\,(w_0,\varepsilon_k)_{0,D}\,\varepsilon_k$ for $t\geq 0$, and set $\widetilde{\mathcal{S}}(t)w_0=e^{-\mu^2\,t}\,\mathcal{S}(t)w_0$ for $t\geq 0$. Next, follow Chapter 3 in [21], to obtain

$$\begin{aligned}\left\|\partial_t^\ell\widetilde{\mathcal{S}}(t)w_0\right\|_{\dot{\mathbf{H}}^p}^2&=\sum_{k=1}^{\infty}\lambda_k^{2p}\left(\partial_t^\ell\widetilde{\mathcal{S}}(t)w_0,\varepsilon_k\right)_{0,D}^2\\&=\sum_{k=1}^{\infty}\lambda_k^{2p}\,(\mu_k+\mu^2)^{2\ell}\left(\widetilde{\mathcal{S}}(t)w_0,\varepsilon_k\right)_{0,D}^2\\&=\sum_{k=1}^{\infty}\lambda_k^{2p}\,(\mu_k+\mu^2)^{2\ell}\,e^{-2\,(\mu_k+\mu^2)\,t}\left(w_0,\varepsilon_k\right)_{0,D}^2,\end{aligned}$$

which yields

$$\left\|\partial_t^\ell\widetilde{\mathcal{S}}(t)w_0\right\|_{\dot{\mathbf{H}}^p}^2\leq\,\widetilde{C}_{\mu,\ell}\,\sum_{k=1}^{\infty}\lambda_k^{2(p+4\ell)}\,e^{-\lambda_k^4\,t}\,(w_0,\varepsilon_k)_{0,D}^2,\tag{A.1}$$

where $\widetilde{C}_{\mu,\ell}:=\left(1+\frac{\mu}{\pi^2}+\frac{\mu^2}{\pi^4}\right)^{2\ell}$. Now, use (A.1), to have

$$\begin{aligned}\int_{t_a}^{t_b}(\tau-t_a)^\beta\left\|\partial_t^\ell\widetilde{\mathcal{S}}(\tau)w_0\right\|_{\dot{\mathbf{H}}^p}^2d\tau&\leq\widetilde{C}_{\mu,\ell}\,\sum_{k=1}^{\infty}\lambda_k^{2(p+4\ell-2\beta)}\left(\int_{t_a}^{t_b}\left[\lambda_k^4(\tau-t_a)\right]^\beta e^{-\lambda_k^4\,\tau}\,d\tau\right)(w_0,\varepsilon_k)_{0,D}^2\\&\leq\widetilde{C}_{\mu,\ell}\,\sum_{k=1}^{\infty}\lambda_k^{2(p+4\ell-2\beta-2)}\left(\int_0^{\lambda_k^4\,(t_b-t_a)}\rho^\beta\,e^{-(\rho+\lambda_k^4t_a)}\,d\rho\right)(w_0,\varepsilon_k)_{0,D}^2\\&\leq\widetilde{C}_{\mu,\ell}\left(\int_0^\infty\rho^\beta\,e^{-\rho}\,d\rho\right)\sum_{k=1}^{\infty}\lambda_k^{2(p+4\ell-2\beta-2)}\,(w_0,\varepsilon_k)_{0,D}^2,\end{aligned}$$

which yields

$$\int_{t_a}^{t_b}(\tau-t_a)^\beta\left\|\partial_t^\ell\widetilde{\mathcal{S}}(\tau)w_0\right\|_{\dot{\mathbf{H}}^p}^2d\tau\leq\,\widetilde{C}_{\beta,\ell,\mu}\,\|w_0\|_{\dot{\mathbf{H}}^{p+4\ell-2\beta-2}}^2,\tag{A.2}$$

where $\widetilde{C}_{\beta,\ell,\mu}=\widetilde{C}_{\mu,\ell}\int_0^\infty x^\beta\,e^{-x}\,dx$. Observing that $\partial_t^\ell\mathcal{S}(t)w_0=e^{\mu^2\,t}\sum_{m=0}^{\ell}\binom{\ell}{m}\,\mu^{2(\ell-m)}\,\partial_t^m\widetilde{\mathcal{S}}(t)w_0$, and using (A.2), we conclude that

$$\int_{t_a}^{t_b}(\tau-t_a)^\beta\left\|\partial_t^\ell\mathcal{S}(\tau)w_0\right\|_{\dot{\mathbf{H}}^p}^2d\tau\leq\,e^{2\,\mu^2\,T}\,C_{\beta,\ell,\mu}\,\sum_{m=0}^{\ell}\|w_0\|_{\dot{\mathbf{H}}^{p+4m-2\beta-2}}^2$$

which yields (2.16) with $\mathcal{C}_{\beta,\ell,\mu,\mu T}=C_{\beta,\ell,\mu}\,e^{2\,\mu^2\,T}\,\ell$. □